THE ASTROPHYSICAL JOURNAL, 416:331–354, 1993 October 10

TURBULENT CONVECTION WITH OVERSHOOTING: REYNOLDS STRESS APPROACH. II.

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Received 1992 November 6; accepted 1993 April 21

ABSTRACT

We derive a new nonlocal model for *turbulent convection* which incorporates recent advances from laboratory, planetary, and numerical simulation of turbulence, and we show how previous prototypic models can be recovered. *The new model is given by five coupled differential equations, equations (81)–(84) and (100), that yield: convective flux, temperature variance, turbulent kinetic energy in the z-direction, total turbulent kinetic energy, and (rate of) dissipation of kinetic energy.* The solution of these five equations yields all the turbulent quantities as a function of the temperature gradient. The latter is then obtained by solving the *flux conservation law*, equation (96), which we derive anew to account for the kinetic energy flux. The main features of the new model are as follows.

1. Nonlocality.—This basic feature is represented by the third-order moments that enter the governing equations (81)–(84) and (100). In all nonlocal models thus far, these moments were treated with the diffusion approximation. Since the latter yields incorrect results in the case of the convective boundary layer, we avoid it. We derive the dynamic equations for all the third-order moments and solve them analytically.

2. Gravity waves, stable stratification.—The fundamental feature of the overshooting (OV) region is that the flow is stably stratified, $\nabla - \nabla_{ad} < 0$. Under such circumstances, the Kolmogorov spectrum is no longer valid since eddies, working against gravity, lose a fraction of their kinetic energy which goes to generate "gravity waves." To fully account for the appearance of a "buoyancy subrange" $E(k) \sim k^{-3}$ in lieu of the Kolmogorov spectrum $\sim k^{-5/3}$, we adopt a recent model for stably stratified turbulence which has been successfully tested against convective boundary layer data.

3. Dissipation ϵ.—The process of dissipation of turbulent kinetic energy has been neglected for many years, but is now viewed as crucial for a proper quantification of OV. The assumption $\epsilon = 0$ not only violates the energy conservation law, but overestimates the extent of the OV region. When ϵ is included, it is generally computed *locally* with a mixing length l. If the description of l is difficult in the main convective region, it is all the more so in the OV region where the concept of a mixing length loses its physical content. We avoid the use of a mixing length in both the convective and the OV region by introducing a differential equation for the dissipation ϵ, equation (100), which, being nonlocal, accounts for the fact that turbulent kinetic energy created at one point in the flow may be dissipated somewhere else, in accordance with the nonlocal nature of turbulent convection.

4. Pressure forces, anisotropy.—The stably stratified turbulence found in the OV region is experimentally known to be highly anisotropic since negative buoyancy suppresses the eddy vertical motion. Thus, pressure-velocity and pressure-temperature correlations, which help restore isotropy, play a crucial role.

5. The Boussinesq Approximation is avoided.

6. The turbulent kinetic energy flux.—A new flux conservation law, equation (96), is derived which includes the turbulent kinetic energy flux recently found to be up to 50% of the total flux for Sun-like stars.

7. A new hydrostatic equilibrium equation, equation (103), is derived which, in addition to a turbulent pressure, also includes buoyancy effects.

The next step is to couple the new model to a stellar structure code.

Subject headings: convection — stars: interiors — turbulence

1. INTRODUCTION

By its own nature, convective turbulence is a highly nonlocal phenomenon. By contrast, another common type of turbulent flow, shear-generated turbulence, is considerably more local in nature due to the work required by the eddies in working against gravity. The mixing length theory (MLT, Kippenhahn & Weigert 1991), which thus far has been the most widely used model, is by construction a local model and thus cannot be considered a satisfactory theory. Over the years, there has been a succession of models that endeavored to treat convection as a nonlocal phenomenon (for a recent discussion, see Unno, Kondo, & Xiong 1985). Some models used as a starting point the local MLT formalism (Shaviv & Salpeter 1973). Other models account for nonlocality by adding to the MLT formalism a *differential equation* for one turbulent variable, usually taken to be the turbulent kinetic energy in the z-direction (Shaviv & Chitre 1968; Bressan, Bertelli, & Chiosi 1981; Kuhfuss 1986; Umezu 1992). A substantial improvement occurred with the advent of the work of Xiong (1985a, b, 1986) who suggested *three differential equations* for $\overline{w^2}$, $\overline{\theta^2}$, and $\overline{w\theta}$ (w and θ are the velocity and temperature fluctuations). Nonlocality, represented by the third-order moments, was treated with the diffusion approximation, the dissipation ϵ was treated locally with the introduction of a mixing length $l = c_1 H_p$, and turbulence was assumed to be isotropic. Canuto (1992), using the Reynolds stress method, went one step further and suggested *four differential equations* for $\overline{q^2}$, $\overline{w^2}$, $\overline{\theta^2}$, and $\overline{w\theta}$ ($\frac{1}{2}q^2$ is the turbulent kinetic energy) since he considered the more general case of anisotropic flow (stably stratified turbulence is known to be highly anisotropic). However, the major difference with Xiong's model is that the crucial third-order moments are no longer treated with the diffusion approximation since the latter yields incorrect results for the convective planetary boundary layer. Rather, Canuto (1992) derived the full dynamic equations for the third-order moments.

At this point in the development of a nonlocal theory of convection, it would not seem advisable to employ models less complete than the last two, especially in the light of the demonstrably improved results that this type of models provides (Unno et al. 1985; Unno & Kondo 1989).

In spite of their completeness with respect to past models, neither Xiong nor Canuto's models represent as yet the most complete description of nonlocal convection. By revisiting the problem, we have found several physical processes the treatment of which must and can be improved using recent advances in turbulence modeling derived from other fields. Below, we discuss these processes in quantitative terms trying to bring to light their physical content so that a reader can skip the mathematical derivations and go directly to the new equations, equations (81)–(84), (96), and (100).

1.1. *Nonlocality*

In a nonlocal treatment of convection, nonlocality is represented by the third-order moments, $\overline{wq^2}$, $\overline{\theta^2 w}$, $\overline{w^3}$, and $\overline{\theta w^2}$ that enter the equations for the second-order moments $\overline{q^2}$, $\overline{\theta^2}$, $\overline{w^2}$, and $\overline{\theta w}$, as well as the equation for the dissipation ϵ. Physically, the third-order moments represents the "fluxes" of the second-order moments. It has been common practice thus far to use a molecular analogy, i.e., a "*diffusion approximation*" whereby each flux is assumed to take place along the gradient of the variable to be diffused, that is

$$\overline{wq^2} = -\nu_t \frac{\partial}{\partial z} \overline{q^2} , \qquad \overline{\theta^2 w} = -\nu_t \frac{\partial}{\partial z} \overline{\theta^2} , \qquad \overline{w^3} = -\nu_t \frac{\partial}{\partial z} \overline{w^2} , \qquad \overline{\theta w^2} = -\nu_t \frac{\partial}{\partial z} \overline{w\theta} , \tag{1}$$

where $\nu_t \approx wl$ (Stellingwerf 1982, eq. [27]; Kuhfuss 1986, eq. [36]; Xiong 1985b, Appendix).

Recent studies (Finger & Schmidt 1986; Paper I; Canuto et al. 1993) have however shown that equations (1) are severely incomplete since each third-order moment must depend on all second-order moments; in addition, the turbulent viscosity ν_t must be replaced by an effective $\nu_t^{\rm eff}$ that includes the convective flux itself. To obtain the complete expressions, we have constructed the dynamic equations governing the third-order moments and solved them analytically, equations (71)–(76). The solution shows that *each third-order moment is a linear combination of the gradients of all the second-order moments,*

$$(\overline{w^3}, \overline{w^2\theta}, \overline{w\theta^2}, \overline{\theta^3}, \overline{wq^2}, \overline{q^2\theta}) = -A \frac{\partial}{\partial z} \overline{w^2} - B \frac{\partial}{\partial z} \overline{w\theta} - C \frac{\partial}{\partial z} \overline{\theta^2} - D \frac{\partial}{\partial z} \overline{q^2} , \tag{2}$$

where $A, \ldots, D$ play the role of a turbulent viscosity (or diffusivity).

One may have suspected that the diffusion approximation (1) was not correct since only the large, energy-containing eddies are *diffusive* in nature whereas smaller eddies are predominantly *dissipative*. The unreliability of (1) to describe the (downward) flux of turbulent kinetic energy has also emerged as one of the main results of numerical simulations of turbulent convection (Chan & Sofia 1989). Furthermore, in the diffusion approximation (1), each turbulent diffusivity $A, \ldots, D$ is assumed to be only a mechanical turbulent viscosity, i.e., a velocity times a mixing length,

$$A, \ldots, D \approx \nu_t \approx wl , \tag{3a}$$

while the new solution shows that $A, \ldots, D$ are also contributed by buoyancy itself, that is

$$A, \ldots, D \approx \nu_t^{\rm eff} = a\nu_t + b\overline{w\theta} , \tag{3b}$$

where the a and b depend on the specific third-order moment (Appendix B). The effect of the convective flux in equation (3b) is as follows (Fig. 1): in the main convective region where $\overline{w\theta} > 0$, this term increases the turbulent viscosity, whereas in the overshoot region where $\overline{w\theta} < 0$, the new term leads to a lower effective turbulent viscosity. Stated differently, in the convective region ($\nabla - \nabla_{\rm ad} > 0$), the standard expression $\nu_t^{\rm eff} = \nu_t$ *underestimates* the true turbulent viscosity, whereas in the overshooting region ($\nabla - \nabla_{\rm ad} < 0$), it *overestimates* it. The third-order moments are given by equations (71)–(76).

2.2. *Dissipation*

In turbulence modeling, a most critical quantity is ϵ, the rate of dissipation of turbulent kinetic energy (Batchelor 1971). Since there have been some misunderstandings on the physical meaning and the value of ϵ, it may be useful to discuss some of the issues. Many models have used $\epsilon = 0$, which results in a violation of the energy conservation besides leading to an overestimation of the overshooting region. Recent models (e.g., Umezu 1992) have emphasized the need to include ϵ but compute it locally.

CONVECTIVE REGION	OVERSHOOTING REGION
$\overline{w\theta} > 0$	$\overline{w\theta} < 0$
$\nabla - \nabla_{\rm ad} > 0$	$\nabla - \nabla_{\rm ad} < 0$
$\overline{\theta^2} \rightarrow \overline{w^2}$	$\overline{w^2} \rightarrow \overline{\theta^2}$

FIG. 1.—A convective regime is characterized by a positive convective flux, a temperature gradient larger than the adiabatic value, and by the physical transformation of temperature fluctuations into velocity fluctuations. The overshooting region is characterized by a regime where velocity fluctuations feed temperature fluctuations or equivalently, *gravity waves.*

In deriving the equation for $\overline{u_i u_j}$ from the Navier-Stokes equations, we have two terms that depend on the molecular viscosity ν: the first term vanishes as $\nu \to 0$, while the second term ($\equiv \epsilon$) does not vanish even when $\nu \to 0$. This is because ϵ is the product of ν times the mean-square vorticity which diverges as $\nu \to 0$, yielding a finite ϵ. This can be seen as follows. The two terms are

$$\nu\left(\overline{u_j \frac{\partial^2 u_i}{\partial x_k^2}} + \overline{u_i \frac{\partial^2 u_j}{\partial x_k^2}}\right) = \nu \frac{\partial^2}{\partial x_k^2} \overline{u_i u_j} - 2\nu \overline{\left(\frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k}\right)} = \nu \frac{\partial^2}{\partial x_k^2} \overline{u_i u_j} - \epsilon_{ij} , \tag{4a}$$

where the tensor ϵ_{ij} is defined as

$$\epsilon_{ij} \equiv 2\nu \overline{\left(\frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k}\right)} . \tag{4b}$$

As $\nu \to 0$, the first term in the last expression (4a) vanishes, while ϵ_{ij} does not vanish *since as ν decreases*, the mean-square vorticity $\sim\overline{(\partial u_i/\partial x_j)^2}$ *increases so that the product (4b) remains finite.* It has been customary for many years to assume that ϵ_{ij} is diagonal since the scales involved in the dissipation process are sufficiently small to be considered isotropic. Recently, this assumption has been found to be incorrect both experimentally and through large eddy simulation techniques (Feiereisen et al. 1982; Browne, Antonia, & Shah 1987). Thus, we employ a more general model of the form (b_{ij} represents the degree of anisotropy, see eq. [12c])

$$\epsilon_{ij} = \frac{2}{3} \epsilon \delta_{ij} + (1 - F^{1/2})\epsilon \frac{b_{ij}}{e} , \tag{4c}$$

where F is zero for two-dimensional turbulence and $F^{1/2} \sim 0.8$ for shear turbulence (the definition of F is given in Appendix A, Eq. [A10]).

Let us now consider how taking $\epsilon = 0$ violates global energy conservation. This can be seen as follows. Consider the equation for the turbulent kinetic energy, equation (35e, Paper I) or equation (41) below

$$\frac{\partial}{\partial t} \frac{1}{2} \overline{q^2} + \frac{\partial}{\partial z}\left(\frac{1}{2} \overline{q^2 w} + \overline{pw}\right) = g\alpha \overline{w\theta} - \epsilon . \tag{5a}$$

The first term on the right-hand side represents the source or production by buoyancy ($P = g\alpha\overline{w\theta}$), while ϵ represents dissipation. Equation (5a) can thus be written as

$$\text{Time evolution} + \text{Diffusion} = \text{Source} - \text{Sink} . \tag{5b}$$

Taking a volume average (denoted by a bar) and a stationary case, one obtains a relation stating that production equals dissipation, namely

$$\bar{P} = \bar{\epsilon} . \tag{5c}$$

Thus, taking $\epsilon = 0$ violates energy conservation since ϵ represents the sink of energy available to the system (Marcus 1982). It is important to stress that it is the *only sink* since the nonlinear interactions conserve energy, and thus the energy input at the largest scales arrives unaltered at the smaller scales where it is dissipated into heat by the only processes operating at those scales, i.e., molecular processes.

Numerical simulations of turbulent convection have further clarified the role of ϵ. Figure 19 of Chan & Sofia (1989) shows that ϵ is of the same order as the buoyancy work, but that it exceeds it near the upper and lower boundaries of the convective zone so as to satisfy the integral relation (5c). See Figure 2.

In § 13 we discuss models that take $\epsilon = 0$ (Roxburgh 1978, 1989; Bressan et al. 1981; Bertelli et al. 1986; Zahn 1991) as well as models that include ϵ *but compute it locally* (Shaviv & Chitre 1968; Antia and Chitre 1993; Kuhfuss 1986; Umezu 1992; and Xiong 1986). The local parameterization employed by Maeder (1975) and Doom (1985) is also discussed in § 13.

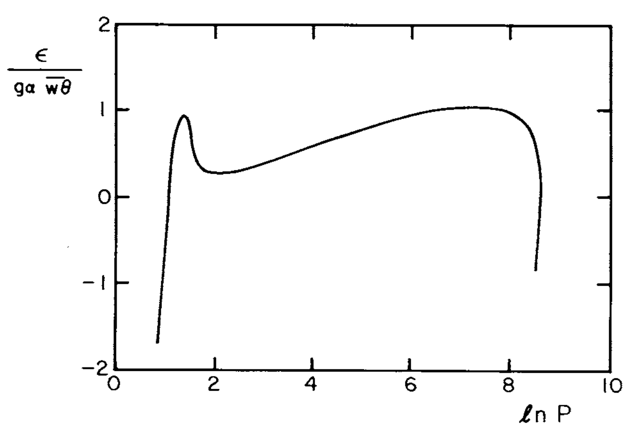

FIG. 2.—The ratio of the local values of the rate of dissipation of kinetic energy ($\equiv\epsilon$) to production ($P = g\alpha\overline{w\theta}$), as from a large eddy simulation of turbulent convection in a solar-like star (K. L. Chan, private communication, 1992).

Finally, we show that ϵ does not depend on ν and that ϵ_θ (the rate of dissipation of temperature variance) does not depend on χ (the thermometric conductivity). Consider the definitions of the turbulent kinetic energy e and of the temperature variance $\overline{\theta^2}$

$$e = \int_{\pi/l}^{\infty} E(k)dk\,, \qquad \overline{\theta^2} = \int_{\pi/l}^{\infty} G(k)dk\,, \tag{6}$$

where l is the size of the largest eddy and $E(k)$ and $G(k)$ are spectral functions for the velocity and temperature variances, respectively. In the case of *unstably stratified turbulent convection,*

$$\frac{\partial T}{\partial z} - \left(\frac{\partial T}{\partial z}\right)_{\rm ad} < 0 \quad \text{or} \quad \nabla - \nabla_{\rm ad} > 0\,, \tag{7}$$

the eddies are described by Kolmogorov inertial spectra (Ko = 1.6 ± 0.02 and Ba = 1.34 ± 0.02 are the Kolmogorov and Batchelor constants, Andreas 1987; Monin & Yaglom 1971, 1975)

$$E(k) = {\rm Ko}\, \epsilon^{2/3} k^{-5/3}\,, \qquad G(k) = {\rm Ba}\, \epsilon_\theta \epsilon^{-1/3} k^{-5/3} \tag{8}$$

Combining equations (8) and (6), one derives

$$\epsilon = \frac{e^{3/2}}{l_\epsilon}\,, \qquad \epsilon_\theta = \frac{1}{2} c_\theta \epsilon \frac{\overline{\theta^2}}{e}\,, \tag{9a}$$

where $l_\epsilon = l c_\epsilon^{-1}$ and

$$c_\epsilon = \pi\left(\frac{2}{3}{\rm Ko}^{-1}\right)^{3/2}\,, \qquad \frac{1}{2} c_\theta = c_\epsilon \frac{\rm Ko}{\rm Ba}\,. \tag{9b}$$

Equations (9) show that ϵ and ϵ_θ do not depend on ν and χ. Viscosity enters only in the expression for the scale l_d at which dissipation occurs. In fact, consider the Fourier transform of equation (4b),

$$\epsilon = 2\nu \int k^2 E(k)dk\,, \tag{10a}$$

and use equation (8). Integrating between 0 and $k_d \sim l_d^{-1}$, we obtain

$$l_d = (\nu^3 \epsilon^{-1})^{1/4} \sim \nu^{3/4}\,, \tag{10b}$$

which states that for a given energy input P, the dissipation of that energy occurs at scales l_d that become progressively smaller as the viscosity of the system decreases. Thus, a low-viscosity system like a star (Massaguer 1990) does not imply that ϵ is zero, but rather that dissipation occurs at scales considerably smaller than in a more viscous system.

In summary, *the value of ϵ is determined by the large scales while the length scales at which ϵ is dissipated is determined by the viscosity ν.*

We shall propose both a local and a nonlocal model to compute ϵ, § 11.

1.3. *The Effect of Pressure Anisotropy*

Like other physical variables, pressure can be decomposed into two components: an ensemble average part and a fluctuating (turbulent) part. In astrophysical convection, pressure fluctuations are usually neglected except for the occasional addition of a turbulent pressure in the hydrostatic equilibrium equation (see § 12). However, in the context of OV, turbulent pressure is perhaps the least important of all the pressure-induced contributions. The distinguishing feature of OV is the appearance of a negative convective flux. Since *negative buoyancy greatly suppresses the vertical motion of the eddies,* the OV region *exhibits a degree of anisotropy larger than the main convective region.* Since pressure forces in the Navier-Stokes equations specifically deal with anisotropy, they must be fully accounted for in any treatment of OV. (We may add that in the strongly convective planetary boundary layer [Holtslag & Moeng 1991, Fig. 1], the pressure-temperature correlations are found to be even larger than the third-order moment $\overline{\theta w^2}$ which is an integral part of the nonlocal character of convection.)

Turbulence modeling over four decades has demonstrated that it is very difficult to account for pressure fluctuations. Work in this field began 40 yr ago with a classic paper by Rotta (1951) and continues to date (Shih & Lumley 1984; Shih and Shabbir 1992; Speziale 1991; Speziale, Gatski, & Sarkar 1992; Taulbee 1992). The essence of the problem can be stated as follows. The Reynolds splitting of the Navier-Stokes equations leads to the following equation for the fluctuating velocity u_i (equation [25])

$$\frac{\partial}{\partial t} u_i + u_j \frac{\partial}{\partial x_j} u_i = -\frac{\partial p}{\partial x_i} + \text{buoyancy} + \text{stresses}\,. \tag{11}$$

The question then arises as how to treat the pressure term. We shall discuss four points:

First, consider the equation for the Reynolds stress $\overline{u_i u_j}$. Pressure forces contribute the pressure-strain tensor

$$\Pi_{ij} = \overline{u_i \frac{\partial p}{\partial x_j}}\,, \tag{12a}$$

which must be modeled. We shall discuss such a tensor in § 5.

In the case of the kinetic energy, the modeling of equation (12a) is simpler since one needs only the trace

$$\Pi_{ii} = \frac{\partial}{\partial x_i} \overline{pu_i} . \tag{12b}$$

In the case of homogeneous turbulence, $\Pi_{ii} = 0$. In a inhomogeneous case, Π_{ii} is not zero but it can be absorbed in the already existing third-order moment originating from the nonlinear term in equation (11); see equations (84) and (47). This applies to a set of models discussed in § 13.

Second, consider the equation for w^2, where w is the velocity in the z-direction. In this case, the effect of pressure cannot be accounted for by the above renormalization alone. This is because pressure forces tend to isotropize the energy among the different components of the kinetic energy u^2, v^2, and w^2. Thus, one expects Π_{33} to contribute a term that *reduces the anisotropy in direct proportion to the degree of anisotropy itself.* If one characterizes the degree of anisotropy by the tensor,

$$b_{ij} = \overline{u_i u_j} - \tfrac{2}{3} e\, \delta_{ij} . \tag{12c}$$

Rotta (1951) first suggested that

$$\Pi_{ij} = \tau^{-1} b_{ij} , \tag{12d}$$

where τ is a time scale to be defined later. Thus we expect that the pressure-velocity correlation contributes terms like

$$\frac{\partial}{\partial t} \overline{w^2} + \frac{\partial}{\partial z} \overline{w^3} = -\tau^{-1}\left(\overline{w^2} - \frac{1}{3} \overline{q^2}\right) + \cdots . \tag{12e}$$

The "return to isotropy" represented by the first term becomes all the more significant, the more anisotropic the flow is. In fact, if w^2 is large, the term acts like a sink, thus reducing the large initial value; if w^2 is too small, the term acts like a source to increase it. Laboratory data on stably stratified turbulence (Webster 1964) have shown that the ratio $\overline{q^2}/\overline{w^2}$ grows from 3 (isotropy) to 6 as the Richardson number increases from 0 to 0.4 (see also Itsweire & Helland 1989; Holt, Koseff, & Ferziger 1992).

Third, consider the equation for the convective flux obtained by multiplying equation (11) by θ (fluctuating temperature) and averaging; see equation (34) below. The pressure term generates the term

$$\Pi_i^\theta = \overline{\theta \frac{\partial p}{\partial x_i}} . \tag{13}$$

While the complete structure of this vector will be discussed later, equation (53), the same type of physical argument presented earlier suggests that equation (13) contributes terms of the form

$$\frac{\partial}{\partial t} \overline{w\theta} + \frac{\partial}{\partial z} \overline{w^2\theta} = -\tau^{-1} \overline{w\theta} + \cdots . \tag{14}$$

In the main convective region, where $\overline{w\theta} > 0$, the pressure term acts like a sink for the convective flux $\overline{w\theta}$ (in fact, the vector $\overline{u_i \theta}$ must vanish in an isotropic case), whereas in the OV region, where $\overline{w\theta} < 0$ the pressure term acts like a source and tends to restore it.

Fourth, from a turbulence theory viewpoint, we recall that the neglect of the pressure terms allows the third-order terms to grow too rapidly, which in turn requires unphysical clipping. This phenomenon is mirrored in the Fourier space analysis (Orszag 1977), where it was found that once one uses the Gaussian approximation for the fourth-order correlations (Paper I, eq. [41]), there is insufficient damping, a problem that is usually solved with the introduction of an *irreversible damping* represented by the pressure terms. This procedure is at the heart of the eddy-damped quasi-normal Markovian (EDQNM) model that has proved very successful in the treatment of isotropic turbulence (Lesieur 1991).

In conclusion, since *OV occurs in a stably stratified regime characterized by a high degree of anisotropy*, the pressure correlations become particularly relevant since they are the only terms in the Navier-Stokes equations that deal with this phenomenon. In our previous treatment (Paper I), we used expressions for Π_{ij} and Π_i^θ first pioneered by Lumley and collaborators who expanded equations (12a) and (13) in power of both b_{ij} and $\overline{u_i \theta}$ and retained only the linear terms. Recently, Shih & Lumley (1985), Fu, Launder, & Tselepidakis (1987), Shih & Shabbir (1992), Speziale (1992), and Speziale et al. (1992) have generalized the original expressions to account for nonlinear terms. These new expressions will be employed here to assure a better treatment of the anisotropic turbulence that characterizes the OV region.

Finally, even though it is not explicitly stated, the first pressure term in equation (14) is responsible for the linear term in the phenomenological equations for the convective flux (Gough 1976; Ulrich 1976) discussed in § 13.

1.4. *Stable Stratification, Gravity Waves*

Are the expressions for the dissipations ϵ and ϵ_θ given by equations (9) valid in general? The answer is no, since the OV region is characterized by *stably stratified turbulence*

$$\frac{\partial T}{\partial z} - \left(\frac{\partial T}{\partial z}\right)_{\rm ad} > 0 \quad \text{or} \quad \nabla - \nabla_{\rm ad} < 0 , \tag{15}$$

where the convective flux is negative, $\overline{w\theta} < 0$. In this situation, eddies working against gravity lose kinetic energy which feeds density fluctuations (*gravity waves*). This loss of energy by the eddies violates the premises for the validity of the Kolmogorov spectrum. Consequently, equations (8) are no longer valid in the OV region. We shall see in § 6 that equations (9) must be generalized to become functions of the turbulent kinetic energy e

$$c_\epsilon \to C_\epsilon(e) , \qquad c_\theta \to C_\theta(e) , \tag{16}$$

and the problem is then to construct the functions $C_\epsilon(e)$ and $C_\theta(e)$ so as to take into account the different physical nature of turbulence in a stably stratified case. We use recent results pertaining to stably stratified turbulence to compute the functions C_ϵ and C_θ.

1.5. *Kinetic Energy Flux*

The exact dynamic equations indicate that the complete flux conservation is

$$F_{\rm ext} = F_r + F_c + F_{\rm KE} , \tag{17}$$

where $F_{\rm KE} = \frac{1}{2}\rho\overline{q^2 w}$ is the flux of the turbulent kinetic energy which has been infrequently included in stellar structure calculations. However, recent numerical simulations have shown that in solar-type stars, $F_{\rm KE}$ can be as large as 50% of the total flux (Chan & Gigas 1992). To include $F_{\rm KE}$, we must extend the formalism beyond the Boussinesq approximation.

1.6. *Hydrostatic Equilibrium Equation*

We find that turbulence alters the standard hydrostatic equation in two ways: the mechanical part of turbulence contributes a turbulent pressure while buoyancy contributes a small term.

1.7. *Thin-Layer Approximation*

It is customary to assume that for a variable A (Spiegel & Veronis 1960)

$$\frac{\partial A}{\partial z} \gg A H_p^{-1} \qquad \text{or} \qquad d \ll H_p , \tag{18}$$

where d is the extent of the convective region. Equation (18) is valid, for example, within the Earth's convective boundary layer, where $d \sim 1$ km and $H_p \sim 7$ km, but becomes invalid in stars where d may encompass several pressure scale heights H_p. In this paper, we remove equation (18) by removing the Boussinesq approximation.

2. THE BASIC EQUATIONS

The basic equations describing a compressible fluid of total density $\tilde\rho$, pressure $\tilde p$, temperature $\tilde T$, velocity v_i, kinematic viscosity ν, and thermal conductivity K ($\equiv c_p \rho\chi$) are given by ($d/dt \equiv \partial/\partial t + v_j \partial/\partial x_j$):

$$\tilde\rho \frac{d}{dt} v_i = -\frac{\partial \tilde p}{\partial x_i} - g_i \tilde\rho + \nu\tilde\rho \frac{\partial^2}{\partial x_j^2} v_i - 2\tilde\rho \epsilon_{ijk} v_k \Omega_j , \tag{19}$$

$$\tilde\rho c_p \frac{d\tilde T}{dt} = \frac{d\tilde p}{dt} + K \frac{\partial^2 \tilde T}{\partial x_j^2} + \mu\left[\frac{\partial^2}{\partial x_i\,\partial x_j} v_i v_j + \left(\frac{\partial v_i}{\partial x_j}\right)^2\right] + c_p \tilde\rho Q , \tag{20}$$

where $c_p \tilde\rho Q$ is the gradient of an external flux, $\mu = \tilde\rho\nu$, ϵ_{ijk} is the antisymmetric tensor, and Ω is the angular velocity. With respect to Paper I, we have added the viscosity term on the right-hand side of the temperature equation to account for the contribution of viscous dissipation to the generation of entropy. This term, which gives rise to ϵ/c_p in equation (26) below, is found to be instrumental for a correct treatment of the complete flux conservation law. In addition, we assume a perfect gas law $\tilde p = R\tilde\rho\tilde T$. First, we split the variables into an average and a fluctuating part

$$\tilde p = P + p , \qquad \tilde T = T + \theta , \qquad \tilde\rho = \rho + \rho' , \qquad v_i = U_i + u_i . \tag{21}$$

Here P, T, ρ, and U_i represent the average fields; the fluctuating components have zero average

$$\overline{p} = \overline{\theta} = \overline{\rho'} = \overline{u_i} = 0 . \tag{22}$$

Assuming further that $P = R\rho T$, and neglecting second-order quantities, we derive for ρ' the relation

$$\frac{\rho'}{\rho} = -\alpha\theta + \frac{p}{P} , \qquad \alpha \equiv \frac{1}{T} . \tag{23}$$

In the standard Boussinesq approximation (Paper I, eq. [21b]), the second term with the fluctuating pressure p/P is absent. The physical interpretation of the terms arising from its inclusion will be discussed shortly. Inserting equations (21)–(23) into equations (19) and (20), and following the procedures outlined in Paper I, we obtain the dynamical equations for the mean and fluctuating variables ($D/Dt \equiv \partial/\partial t + U_i \partial/\partial x_i$, $\lambda_i = g_i \alpha$, $g_i = 0, 0, g$):

$$\frac{DU_i}{Dt} = -\left(g_i + \frac{1}{\rho}\frac{\partial P}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\overline{u_i u_j} + N_1^i - 2\tilde{\rho}\epsilon_{ijk} v_k \Omega_j \,, \tag{24}$$

$$\rho \frac{Du_i}{Dt} = -\rho u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} - \rho \frac{\partial}{\partial x_j}(u_i u_j - \overline{u_i u_j}) + \lambda_i \rho\theta + \nu\rho \frac{\partial^2 u_i}{\partial x_j^2} + N_2^i - 2\tilde{\rho}\epsilon_{ijk} v_k \Omega_j \,, \tag{25}$$

$$\frac{DT}{Dt} = \chi \frac{\partial^2 T}{\partial x_j^2} - \frac{\partial}{\partial x_j}\overline{u_j\theta} + \frac{\epsilon}{c_p} + \frac{1}{c_p} U_j \frac{\partial P}{\partial x_j} + Q + N_3 \,, \tag{26}$$

$$\frac{D\theta}{Dt} = -u_j \frac{\partial T}{\partial x_j} - \frac{\partial}{\partial x_j}(u_j\theta - \overline{u_j\theta}) + \chi \frac{\partial^2 \theta}{\partial x_j^2} + \frac{\nu}{c_p}\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2 - \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}\right] + N_4 \,, \tag{27}$$

where we have taken the $\nu \to 0$ limit wherever possible. The non-Boussinesq terms Ns are given by

$$\rho N_1^i \equiv -\alpha \overline{\theta \frac{\partial p}{\partial x_i}} + \overline{\frac{p}{P}\frac{\partial p}{\partial x_i}} \,, \tag{28}$$

$$N_2^i \equiv -\left(g_i + \frac{1}{\rho}\frac{\partial P}{\partial x_i}\right)\alpha\rho\theta + \frac{p}{P}\frac{\partial P}{\partial x_i} - \alpha\left(\theta \frac{\partial p}{\partial x_i} - \overline{\theta \frac{\partial p}{\partial x_i}}\right) + \frac{1}{P}\left(p \frac{\partial p}{\partial x_i} - \overline{p \frac{\partial p}{\partial x_i}}\right) , \tag{29}$$

$$\rho c_p N_3 \equiv U_j \Lambda_j + \overline{u_j \frac{\partial p}{\partial x_j}} + \left(\alpha\overline{\theta u_j} - \frac{1}{P}\overline{p u_j}\right)\frac{\partial P}{\partial x_j} + \alpha\overline{\theta u_j \frac{\partial p}{\partial x_j}} - \frac{1}{P}\overline{p u_j \frac{\partial p}{\partial x_j}} \,, \tag{30}$$

with

$$\Lambda_j \equiv \alpha \overline{\theta \frac{\partial p}{\partial x_j}} - \frac{1}{P}\overline{p \frac{\partial p}{\partial x_j}} \,, \tag{31}$$

$$c_p \rho N_4 \equiv U_j \Delta_j + u_j \frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}(p u_j - \overline{p u_j}) + \left[\alpha(\theta u_i - \overline{\theta u_i}) - \frac{1}{P}(p u_i - \overline{p u_i})\right]\frac{\partial P}{\partial x_i} + \alpha\left(u_i \theta \frac{\partial p}{\partial x_i} - \overline{u_i \theta \frac{\partial p}{\partial x_i}}\right) - \frac{1}{P}\left(u_i p \frac{\partial p}{\partial x_i} - \overline{u_i p \frac{\partial p}{\partial x_i}}\right) , \tag{32}$$

and with

$$\Delta_i \equiv \frac{\partial p}{\partial x_i} - \left(\frac{p}{P} - \alpha\theta\right)\frac{\partial P}{\partial x_i} + \alpha\left(\theta \frac{\partial p}{\partial x_i} - \overline{\theta \frac{\partial p}{\partial x_i}}\right) - \frac{1}{P}\left(p \frac{\partial p}{\partial x_i} - \overline{p \frac{\partial p}{\partial x_i}}\right) . \tag{33}$$

To understand the physical meaning of these new terms, compare the third term in N_3 with the second term in equation (26). The ratio is of the order of d/H_p which, within the thin-layer approximation, was neglected. Analogous arguments can be applied to terms in N_2 and N_4.

3. SECOND-ORDER MOMENTS

Following the procedure outlined in Paper I, we derive the following results. For simplicity, we shall not consider the mean flow U_i. For the convective flux $\overline{u_i\theta}$ we obtain, in lieu of equation (34a, Paper I), the equation

$$\frac{D}{Dt}\overline{u_i\theta} + \frac{\partial}{\partial x_j}\overline{\theta u_i u_j} = -\overline{u_i u_j}\frac{\partial T}{\partial x_j} + \lambda_i \overline{\theta^2} - \Pi_i^\theta + \eta_i + C_i - 2\epsilon_{ijk}\Omega_j \overline{u_k \theta} \,, \tag{34}$$

where the pressure correlation term is defined as (see Paper I, eq. [34b])

$$\Pi_i^\theta \equiv \overline{\theta \frac{\partial p}{\partial x_i}} \,, \tag{35}$$

and the new term C_i is defined by

$$C_i \equiv \overline{N_2^i \theta} + \overline{\rho N_4 u_i} + \frac{\rho}{c_p}\overline{u_i \epsilon} \,, \tag{36a}$$

with (Paper I, eq. [37f])

$$\overline{u_i \epsilon} = \tau^{-1}\overline{q^2 u_i} \,. \tag{36b}$$

Next, consider the equation for the temperature variance $\overline{\theta^2}$. In lieu of equation (35a, Paper I), we have

$$\frac{D\overline{\theta^2}}{Dt} + \frac{\partial}{\partial x_i}\overline{u_i\theta^2} = -2\overline{u_i\theta}\frac{\partial T}{\partial x_i} + \chi\frac{\partial^2\overline{\theta^2}}{\partial x_j^2} - 2\epsilon_\theta + C^\theta , \tag{37}$$

where

$$\frac{1}{2}C^\theta \equiv \overline{N_4\theta} + \frac{1}{c_p}\overline{\epsilon\theta} , \tag{38a}$$

with equation (38e, Paper I)

$$\overline{\epsilon\theta} = \tau^{-1}\overline{q^2\theta} . \tag{38b}$$

Next, consider the equation for the Reynolds stress $\overline{u_iu_j}$. We derive

$$\frac{D}{Dt}\overline{u_iu_j} + \frac{\partial}{\partial x_k}D_{ijk} = \lambda_i\overline{u_j\theta} + \lambda_j\overline{u_i\theta} - \Pi_{ij} - \epsilon_{ij} + C_{ij} - 2\Omega_{ij} , \tag{39}$$

where ϵ_{ij} is defined in equation (4b) and (we use the notation $a_{,j} \equiv \partial a/\partial x_j$)

$$D_{ijk} \equiv \overline{u_iu_ju_k} , \qquad \Omega_{ij} \equiv (\epsilon_{ipk}\overline{u_ju_k} + \epsilon_{jpk}\overline{u_iu_k})\Omega_p , \qquad \Pi_{ij} \equiv \overline{u_ip_{,j}} + \overline{u_jp_{,i}} , \qquad C_{ij} \equiv \overline{N_2^iu_j} + \overline{N_2^ju_i} . \tag{40}$$

Clearly, the third-order moments D_{ijk} and Π_{ij} represent the flux of $\overline{u_iu_j}$ and the pressure-strain correlation. Finally, by taking the trace of equation (39) one obtains the equation for the turbulent kinetic energy per unit mass $e = \frac{1}{2}\overline{u_iu_i}$,

$$\frac{De}{Dt} + \frac{1}{2}\frac{\partial}{\partial x_k}D_{iik} = \lambda_i\overline{u_i\theta} - \frac{1}{2}\Pi_{ii} - \epsilon + \frac{1}{2}C_{ii} - \Omega_{ii} . \tag{41}$$

4. EVALUATION OF THE C TERMS

We shall now compute the C terms arising from the non-Boussinesq approximation. Making use of several result of Paper I, we derive (the mean flow contribution can easily be evaluated if necessary):

$$\begin{aligned}\rho^{-1}C_3 = &-\frac{g}{c_p}\left(\overline{w^2} + c_p\frac{1}{P}\overline{p\theta}\right)\left(1 - \frac{g_*}{g}\right) - 2c_8\tau^{-1}\alpha\overline{w\theta^2} - c_{11}g\alpha^2\overline{\theta^3} \\ &+ (3\tau_3c_p)^{-1}\overline{w^3} + g\alpha c_p^{-1}\left(c_{11} - 1 + \frac{g_*}{g}\right)\overline{\theta w^2} - g_*\alpha\overline{\theta^2} + (\tau c_p)^{-1}\overline{q^2w} ,\end{aligned} \tag{42}$$

where

$$g_* \equiv g\left(1 + \frac{1}{g\rho}\frac{\partial P}{\partial z}\right) \approx \frac{\partial}{\partial z}\overline{w^2} = \rho^{-1}\frac{\partial}{\partial z}p_t , \tag{43a}$$

$$\overline{p\theta} = -C_p^\theta\rho\overline{q^2\theta} . \tag{43b}$$

In equation (43a) we have used equation (24) without N_1^i, and p_t is the turbulent pressure. Similarly, we derive

$$\frac{1}{2}C^\theta = -\frac{g}{c_p}(\overline{w\theta} + \alpha\overline{\theta^2w})\left(1 - \frac{g_*}{g}\right) - (3c_9 - 1)(\tau c_p)^{-1}\overline{q^2\theta} , \tag{44}$$

and

$$\frac{1}{2\rho}C_{ii} = -\frac{g}{P}\overline{pw}\left(1 - \frac{g_*}{g}\right) + 3c_9\tau^{-1}\alpha\overline{q^2\theta} - g_*\alpha\overline{\theta w} , \tag{45}$$

$$\frac{1}{2\rho}C_{33} = -\frac{g}{P}\overline{pw}\left(1 - \frac{g_*}{g}\right) - \frac{2}{3}c_{11}g\alpha^2\overline{w\theta^2} - c_8\tau^{-1}\alpha\overline{w^2\theta} + \frac{1}{3}(c_8 + 3c_9)\tau^{-1}\alpha\overline{q^2\theta} - g_*\alpha\overline{\theta w} , \tag{46}$$

where, in analogy with the second of equation (43), we take

$$\overline{pw} = -C_p^w\rho\overline{q^2w} . \tag{47}$$

The numerical coefficients in equations (43) and (47) are given in Appendix A. It is important to note that in the Boussinesq limit

$$\frac{1}{2}C^\theta \to -\frac{g}{c_p}\overline{w\theta}\,, \qquad C_3 \to -\rho\frac{g}{c_p}\overline{w^2}\,, \tag{48a}$$

so that the terms $\partial T/\partial z$ in equations (34) and (37) get renormalized to

$$\frac{\partial T}{\partial z} \to \frac{\partial T}{\partial z} + \frac{g}{c_p} = \frac{\partial T}{\partial z} - \left(\frac{\partial T}{\partial z}\right)_{\rm ad} \equiv -\beta(z)\,, \tag{48b}$$

as expected (see also Paper I, eq. [27a] for the definition of β). With relations (42)–(47), we have expressed the functions Cs either in terms of second-order moments, which we solve for, or in terms of third-order moments which we shall express in terms of second-order moments in § 7.

5. PRESSURE CORRELATIONS

First, consider the pressure-temperature correlation term (35). Because Poisson's equation (Paper I, eq. [42]) is linear in p, it has been customary (Lumley 1978) to distinguish three contributions: a *rapid part* due to the interaction between turbulence and the mean flow, a *return-to-isotropy part*, contributed by the turbulence-turbulence interaction and finally, a *buoyancy part*. We write (in this section, the pressure is understood to be in units of the density ρ)

$$\Pi_i^\theta \equiv \Pi_i^\theta(\text{rap. part}) + \Pi_i^\theta(\text{ret. to isotropy}) + \Pi_i^\theta(\text{buoy.})\,. \tag{49}$$

Since in the present treatment we have no mean flow, the first term is zero. For the second term we write

$$\Pi_i^\theta(\text{ret. to isotropy}) = \frac{\partial}{\partial x_i}\overline{p\theta} - \overline{p\frac{\partial\theta}{\partial x_i}}\,, \tag{50a}$$

where

$$\overline{p\theta} = -C_p^\theta\overline{q^2\theta}\,, \tag{50b}$$

$$\overline{p\frac{\partial\theta}{\partial x_i}} = -f_1\tau^{-1}\overline{u_i\theta} + (\nu+\chi)\overline{\frac{\partial\theta}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\,. \tag{50c}$$

The first term in equation (50c) is the so-called *return-to-isotropy* (or slow part), first suggested by Rotta (1951) and τ is the kinetic energy dissipation time scale

$$\tau = \frac{2e}{\epsilon}\,. \tag{51}$$

Lumley (1978), Shih & Lumley (1985), and Shih & Shabbir (1991) have suggested the expression

$$\Pi_i^\theta(\text{buoy.}) = \gamma_1\lambda_i\overline{\theta^2} + \lambda_j Y_{ij}\,, \tag{52a}$$

$$4eY_{ij} = 2\gamma_2\overline{\theta^2}b_{ij} + 4\gamma_3\overline{\theta u_i}\,\overline{\theta u_j} + 2\gamma_4 e^{-1}(b_{ik}\overline{\theta u_j} + b_{jk}\overline{\theta u_i})\overline{\theta u_k} + \gamma_5 e^{-1}\overline{\theta^2}b_{ik}b_{kj}\,, \tag{52b}$$

where f_1 is a dimensionless function given in Appendix A and b_{ij} is given by equation (12c).

Putting these results together, we have the complete expression

$$\Pi_i^\theta = f_1\tau^{-1}\overline{u_i\theta} + \gamma_1\lambda_i\overline{\theta^2} + \lambda_j Y_{ij} + \frac{\partial}{\partial x_i}\overline{p\theta} - (\nu+\chi)\overline{\frac{\partial\theta}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\,. \tag{53}$$

If we neglect the nonlinear terms by taking

$$\gamma_{2,3,4,5} = 0\,, \tag{54}$$

the first two terms of equation (53) reduce to equation (43a, Paper I) if we call $f_1 \equiv 2c_6$ and $\gamma_1 \equiv c_7$. For the reasons explained in the Introduction, the new nonlinear terms provide a more complete description. The dimensionless functions f_1 and γ's are given in Appendix A.

An analogous treatment of Π_{ij} leads to

$$\Pi_{ij} \equiv \Pi_{ij}(\text{rap. part}) + \Pi_{ij}(\text{ret. to isotropy}) + \Pi_{ij}(\text{buoy.})\,, \tag{55}$$

where, as before, the first term is zero. As for the second term, we write

$$\Pi_{ij}(\text{ret. to isotropy}) = \left(\frac{\partial}{\partial x_j}\overline{pu_i} + \frac{\partial}{\partial x_i}\overline{pu_j}\right) - (\overline{pu_{i,j}} + \overline{pu_{j,i}})\,, \tag{56a}$$

where

$$\overline{pu_i} \equiv -C_p^w \overline{q^2 u_i}\,, \tag{56b}$$

$$(\overline{pu_{i,j}} + \overline{pu_{j,i}}) \equiv -2c_4 \tau^{-1} b_{ij}\,. \tag{56c}$$

Finally,

$$\Pi_{ij}(\text{buoy.}) = (1-\beta_5)\lambda_k B_{ij}^k + \lambda_k \Delta_{ij}^k\,, \tag{57a}$$

where

$$B_{ij}^k \equiv \delta_{ik}\overline{\theta u_j} + \delta_{jk}\overline{\theta u_i} - \tfrac{2}{3}\delta_{ij}\overline{\theta u_k}\,, \tag{57b}$$

and

$$\Delta_{ij}^k \equiv -\tfrac{2}{3}(1+4\beta_5)A_{ij}^k + \tfrac{2}{3}(1-\tfrac{1}{2}\beta_5)C_{ij}^k + e^{-1}(\beta_5 - 1)b_{ij}\overline{\theta u_k} + (\beta_7 + 3\beta_5)D_{ij}^k + \beta_9 E_{ij}^k + \tfrac{3}{2}e^{-2}\beta_5 b_{ij} b_{kp}\overline{\theta u_p}\,, \tag{57c}$$

with

$$2eA_{ij}^k \equiv b_{ik}\overline{\theta u_j} + b_{jk}\overline{\theta u_i} - \tfrac{2}{3}\delta_{ij} b_{pk}\overline{\theta u_p}\,, \tag{57d}$$

$$2eC_{ij}^k \equiv (\delta_{ik} b_{jp} + \delta_{jk} b_{ip} - \tfrac{2}{3}\delta_{ij} b_{kp})\overline{u_p \theta}\,, \tag{57e}$$

$$4e^2 D_{ij}^k \equiv [b_{ik} b_{jp} + b_{jk} b_{ip} - (\delta_{ik} b_{jm} + \delta_{jk} b_{im}) b_{mp}]\overline{u_p \theta}\,, \tag{57f}$$

$$4e^2 E_{ij}^k \equiv (b_{im}\overline{u_j \theta} + b_{jm}\overline{u_i \theta}) b_{mk} - (\delta_{ik} b_{jm} + \delta_{jk} b_{im}) b_{mp}\overline{u_p \theta}\,. \tag{57g}$$

If we neglect the nonlinear terms, only the B_{ij}^k term survives and we recover equations (44 and 44a [Paper I]) provided we call $1 - \beta_5 \equiv c_5$. Finally, we note that

$$\Delta_{ii}^k = 0\,. \tag{57h}$$

Putting these results together, we obtain

$$\Pi_{ij} \equiv 2c_4 \tau^{-1} b_{ij} + (1-\beta_5)\lambda_k B_{ij}^k + \lambda_k \Delta_{ij}^k + \left(\frac{\partial}{\partial x_i}\overline{pu_j} + \frac{\partial}{\partial x_j}\overline{pu_i}\right). \tag{58}$$

The dimensionless functions β_k's and the other constants are given in Appendix A.

6. DISSIPATION RATES ϵ AND ϵ_θ

The calculation that led to the expression of ϵ and ϵ_θ given by equations (9) was based on the assumption that the scales that dissipate kinetic energy and temperature variance, are (a) *isotropic*, (b) *inertial* (describable by a Kolmogorov and Batchelor-Corrsin spectrum, eq. [8]), (c) *passive* (transferring energy without dissipation), and (d) *cascading* energy from the largest to the smallest scales. While (a)–(d) seem justified in the case on unstable stratification (eq. [7]), *all four break down in the case of stable stratification* (eq. [15]) that occurs in the OV region.

Working against gravity, the eddies lose some of their kinetic energy which is transformed into potential energy represented by the $\overline{\theta^2}$ fluctuations (Fig. 1). This can be seen from eq. (82) or Paper I, equation (37), where in the case of stable stratification, the negative flux $\overline{w\theta}$ acts as a source for $\overline{\theta^2}$, while it acts as a sink for the kinetic energy, equation (84) or Paper I, equation (41). Specifically, introducing the potential energy PE,

$$\text{PE} = \tfrac{1}{2} g\alpha\overline{\theta^2}(\partial T/\partial z)^{-1}\,, \tag{59}$$

we derive from equations (37) and (41) ($\text{KE} = \tfrac{1}{2}\overline{u_i u_i}$)

$$\frac{D}{Dt}(\text{KE} + \text{PE}) = -\left(\epsilon + \epsilon_\theta \frac{g\alpha}{\partial T/\partial z}\right) - \frac{1}{2}\Pi_{ii} + \frac{1}{2}C_{ii} + \frac{1}{2}\frac{g\alpha}{\partial T/\partial z}C^\theta\,. \tag{60}$$

First, we note that the convective flux $\overline{w\theta}$ is no longer present: physically, this means that convection is a process that transforms kinetic into potential energy and vice versa and thus, in the total energy balance, is exactly compensated.

In the stably stratified case, the fraction of kinetic energy transformed into potential energy does not, however, simply cascade toward smaller scales and then dissipate through molecular processes. In fact, the latter are too inefficient to dissipate all the kinetic energy extracted from the eddies: the only alternative is a backward flow toward larger scales (Schumann 1987; Dalaudier & Sidi 1987). This process violates (d) above.

Assumption (b) is also violated since the smaller scales can no longer be represented by an inertial spectrum alone since the spectrum now exhibits two ranges, *a buoyancy subrange* and *an inertial subrange* separated by a wavenumber $k_b = \pi/\Delta_b = \pi e^{1/2}/N$, where $N^2 = g\alpha|\partial T/\partial z - (\partial T/\partial z)_{\text{ad}}|$. If $\epsilon(k)$ and $\epsilon_\theta(k)$ represent the fluxes of kinetic energy and temperature fluctuations across a given wavenumber k, the inertial subrange defined by $\epsilon(k) = \text{constant} = \epsilon$ is attained only for wavenumbers $k > k_b$, whereas for $k \le k_b$, $\epsilon(k)$ decreases with k indicating a net loss of energy from the eddies. The *buoyancy subrange* contains gravity waves that are only weakly damped (and thus their effect is maximized), while in the *inertial subrange* gravity waves are strongly damped and a constant flux $\epsilon(k) = \epsilon$ can be attained. Alternatively, one can view the new buoyancy subrange as a "buoyancy modified turbulence," or as "turbulence modified waves" since, as Gargett et al. (1981) have pointed out, one is considering "vertical scales between wavelike motion at larger scales and locally isotropic turbulence at smaller scales." Since the eddies have lost some of their kinetic energy to gravity waves, the energy left to be dissipated by molecular processes is less than in the unstably stratified case and so one concludes that

$$\epsilon(\text{stable}) < \epsilon(\text{unstable}) , \tag{61}$$

and correspondingly

$$\epsilon_\theta(\text{stable}) > \epsilon_\theta(\text{unstable}) , \tag{62}$$

which imply that the functions C_ϵ and C_θ appearing in equation (16) must be such that

$$C_\epsilon(e) < 1 , \qquad C_\theta(e) > 1 . \tag{63}$$

The first turbulence models to account for stable stratification are due to Bolgiano (1959, 1962), Shur (1962), and Lumley (1964). Since there are difficulties with Bolgiano's model (Phillips 1965), we consider Lumley's model which predicts a kinetic energy spectrum $E(k)$ that encompasses both *inertial and buoyancy subranges* depending on the ratio k/k_0, where $k_0 = (N_3/\epsilon)^{1/2}$,

$$E(k) = \text{Ko}\ \epsilon^{2/3}[1 + (k/k_0)^{-4/3}]k^{-5/3} . \tag{64}$$

Equations (64) and the first of equation (8) yield the first of equation (9a) but with

$$c_\epsilon \to C_\epsilon(e) = c_\epsilon(1 - \tfrac{1}{2}\ \text{Ko}\ \pi^{-2}l^2/\Delta_b^2)^{3/2} , \tag{65}$$

which satisfies equation (63). Is the Shur-Lumley model the correct description of stably stratified turbulence? While atmospheric and oceanographic data (Gargett et al. 1981; Gargett 1985, 1989, 1990; Weinstock 1978a, b, 1985; Dalaudier & Sidi 1987) are in qualitative agreement with equation (64) that predicts $E(k) \sim k^{-3}$ for the buoyancy subrange, measured spectra exhibit indices ranging from -2.5 to -3. Morever, Phillips (1965) pointed out that the main assumption underlying equation (64) that $E(k)$ depends on ϵ and k only is actually not valid: the flux of kinetic energy must also depend on the *kinetic energy of the eddies.* Lumley's theory was reviewed by Weinstock (1978a, b, 1980, 1985a, b, 1990) who pointed out, among other things, that a physically complete treatment of a stably stratified flow must explicitly account for the gravity waves that ultimately store the kinetic energy lost by the eddies, and that equation (64) is based on an "unjustified identification of Eulerian and Lagrangian time scales." This implies that $E(k)$ must depend not only on N and k, but also on the kinetic energy itself. In this case, the complete form of the spectrum $E(k, N, e)$ can be expected to be considerably more complex than equation (64). Recently, Canuto & Minotti (1993) and Cheng & Canuto (1993) have used the Weinstock model to derive the following results:

$$\epsilon = \frac{e^{3/2}}{l_\epsilon} , \tag{66}$$

unstable stratification:

$$l_\epsilon/l = c_\epsilon^{-1} = \text{constant} ; \tag{67a}$$

stable stratification:

$$l_\epsilon/l = \{[1 - a_1 x^4(1 + a_2 x^2)^{-2}]^{1/2} - a_3 x^2(1 + a_4 x^2)^{-1}\}^{-3} ; \tag{67b}$$

where

$$x = (-N^2)^{1/2} l e^{-1/2} , \tag{68}$$

$$N^2 \equiv -\alpha g_i\left[\frac{\partial T}{\partial x_i} - \left(\frac{\partial T}{\partial x_i}\right)_{\text{ad}}\right] . \tag{69}$$

At the same time, the dissipation ϵ_θ is computed from

$$\epsilon_\theta = \frac{1}{2} c_\theta \epsilon \frac{\overline{\theta^2}}{e} . \tag{70}$$

The numerical coefficients a_j and c's are given in Appendix A.

In Figure 1, we sketch the main physicial features of the convective and OV regions. In the former, $\nabla - \nabla_{\text{ad}} \geq 0$ (unstably stratified turbulence), the convective flux is positive, $\overline{w\theta} > 0$, density (or temperature) fluctuations feed kinetic energy $\overline{\theta^2} \to \overline{w^2}$, and the dissipation rates ϵ and ϵ_θ are such that ϵ is greater than its counterpart in the OV region, whereas ϵ_θ is less, due to the fact that a fraction of the $\overline{\theta^2}$-stuff goes to feed $\overline{w^2}$ rather than being left to dissipate at the molecular level.

In the OV region, turbulence is stably stratified $\nabla - \nabla_{\text{ad}} \leq 0$, the convective flux is negative $\overline{w\theta} < 0$, eddy kinetic energy is lost in the work against gravity and goes to feed the reservoir of $\overline{\theta^2}$ fluctuations (gravity waves). Since this leaves less kinetic energy to be dissipated, $\epsilon(\text{OV region}) < \epsilon$ and conversely, $\epsilon_\theta(\text{OV region}) > \epsilon_\theta$.

7. THIRD-ORDER MOMENTS

The dynamic equations for the third-order moments were derived in Paper I, equations (61)–(66). In the stationary case, these equations form a set of linear algebraic equations that can be solved analytically. The results are:

$\overline{w^2\theta}$:

$$g\alpha\tau\overline{w^2\theta} = g\alpha\tau A_1 \frac{\partial}{\partial z}\overline{w\theta} + A_2 \frac{\partial}{\partial z}\overline{w^2} + (g\alpha\tau)^2 A_3 \frac{\partial}{\partial z}\overline{\theta^2} + A_4 \frac{\partial}{\partial z}\overline{q^2}\,, \qquad A_k = A_{k1}\tau\overline{w^2} + A_{k2}\, g\alpha\tau^2\overline{w\theta}\,; \tag{71}$$

$\overline{w^3}$:

$$\overline{w^3} = g\alpha B_1 \frac{\partial}{\partial z}\overline{w\theta} + B_2 \frac{\partial}{\partial z}\overline{w^2} + (g\alpha\tau)^2 B_3 \frac{\partial}{\partial z}\overline{\theta^2} + B_4 \frac{\partial}{\partial z}\overline{q^2}\,, \qquad B_k = B_{k1}\tau\overline{w^2} + B_{k2}\, g\alpha\tau^2\overline{w\theta}\,; \tag{72}$$

$\overline{w\theta^2}$:

$$(g\alpha\tau)^2\overline{w\theta^2} = g\alpha\tau C_1 \frac{\partial}{\partial z}\overline{w\theta} + C_2 \frac{\partial}{\partial z}\overline{w^2} + (g\alpha\tau)^2 C_3 \frac{\partial}{\partial z}\overline{\theta^2} + C_4 \frac{\partial}{\partial z}\overline{q^2}\,, \qquad C_k = C_{k1}\tau\overline{w^2} + C_{k2}\, g\alpha\tau^2\overline{w\theta}\,; \tag{73}$$

$\overline{q^2\theta}$:

$$g\alpha\tau\overline{q^2\theta} = g\alpha\tau D_1 \frac{\partial}{\partial z}\overline{w\theta} + D_2 \frac{\partial}{\partial z}\overline{w^2} + (g\alpha\tau)^2 D_3 \frac{\partial}{\partial z}\overline{\theta^2} + D_4 \frac{\partial}{\partial z}\overline{q^2}\,, \qquad D_k = D_{k1}\tau\overline{w^2} + D_{k2}\, g\alpha\tau^2\overline{w\theta}\,; \tag{74}$$

$\overline{q^2w}$:

$$\overline{q^2w} = g\alpha\tau E_1 \frac{\partial}{\partial z}\overline{w\theta} + E_2 \frac{\partial}{\partial z}\overline{w^2} + (g\alpha\tau)^2 E_3 \frac{\partial}{\partial z}\overline{\theta^2} + E_4 \frac{\partial}{\partial z}\overline{q^2}\,, \qquad E_k = E_{k1}\tau\overline{w^2} + E_{k2}\, g\alpha\tau^2\overline{w\theta}\,; \tag{75}$$

$\overline{\theta^3}$:

$$(g\alpha\tau)^3\overline{\theta^3} = g\alpha\tau F_1 \frac{\partial}{\partial z}\overline{w\theta} + F_2 \frac{\partial}{\partial z}\overline{w^2} + (g\alpha\tau)^2 F_3 \frac{\partial}{\partial z}\overline{\theta^2} + F_4 \frac{\partial}{\partial z}\overline{q^2}\,, \qquad F_k = F_{k1}\tau\overline{w^2} + F_{k2}\, g\alpha\tau^2\overline{w\theta}\,. \tag{76}$$

The factors $g\alpha$ and $g\alpha\tau$ that appear on the left-hand sides of equations (71)–(76) were introduced so that all third-order moments have dimensions of (velocity)^3. We note that the commonly used *diffusion approximation* corresponds to taking (Kuhfuss 1986, eq. [34]; Stellingwerf 1982, eq. [27]; Xiong 1985a, b)

$$A_{2,3,4} = 0\,, \quad B_{1,3,4} = 0\,, \quad C_{1,2,4} = 0\,, \quad D_{2,3,4} = 0\,, \quad E_{1,2,3} = 0\,, \quad F_{1,2,4} = 0\,. \tag{77}$$

Recent work has shown, however, that equations (77) fail to reproduce the well-established countergradient phenomenon which has been observed in numerical simulations, laboratory turbulence (Schumann 1987; Finger and Schmidt, 1986), and atmospheric turbulence (Deardorff 1972).

In equation, (71)–(76) the functions $A_k, \ldots, F_k$ play the role of an effective turbulent viscosity or diffusivity (see § 8). The dimensionless functions A_{k1}, A_{k2}, etc., are given in Appendix B.

8. EFFECTIVE TURBULENT VISCOSITY (DIFFUSIVITY)

The third-order moments equations (71)–(76) exhibit a universal structure, namely they are a linear combination of the derivatives of the second-order moments with turbulent viscosities (or diffusivities) $A_k, \ldots, F_k$. The latter also exhibit a universal structure since they all are the combinations of a "*mechanical part*"

$$\nu_t \sim \tau w^2 \sim l e^{1/2}\,, \tag{78}$$

plus a "*buoyancy part*"

$$g\alpha\tau^2\overline{w\theta}\,, \tag{79}$$

so that the effective turbulent viscosity ν_t^{eff} is given by

$$\nu_t^{\text{eff}} = a\nu_t + b g\alpha\tau^2\overline{w\theta}\,, \tag{80}$$

where a and b are dimensionless functions of the variable $(N\tau)^2$, Appendix B.

Since in a convective regime $\overline{w\theta}$ is positive, the usual approximation $\nu_t^{\text{eff}} = \nu_t$ *underestimates* the true value of the turbulent viscosity, whereas in the OV region, where $\overline{w\theta}$ is negative, $\nu_t^{\text{eff}} = \nu_t$ *overestimates* the true turbulent viscosity. The double approximation represented by equation (77) together with $\nu_t^{\text{eff}} = \nu_t$ has been adopted by Kuhfuss (1986, equation [31]), Stellingwerf (1982, eq. [27]), and Xiong (1985a, b).

9. THE NEW MODEL

Using the previous equations, we can finally write the equations governing the variables of interest. We consider the case of parallel geometry. In the case of spherical geometry, the $\partial(\ldots)/\partial z$ terms in the left-hand sides of equations (81)–(84) become

$r^{-2}\partial/\partial r(r^2\ldots)$, while the $\partial(\ldots)/\partial z$ terms in equations (71)–(76) become $\partial(\ldots)/\partial r$. These equations are the generalizations of equations (57)–(60, Paper I) to include the new effects represented by the terms C's and P^{nl} (for ease of notation, we take $\rho = 1$).

Convective flux $\overline{w\theta}$:

$$\frac{\partial}{\partial t}\overline{w\theta} + \frac{\partial}{\partial z}(\overline{\theta w^2} + \overline{p\theta}) = -\frac{\partial T}{\partial z}\overline{w^2} + (1-\gamma_1)g\alpha\overline{\theta^2} - f_1\tau^{-1}\overline{w\theta} + \frac{1}{2}\chi\frac{\partial^2}{\partial z^2}\overline{w\theta} + C_3 - P_\theta^{nl} \; ; \tag{81}$$

Temperature variance $\overline{\theta^2}$:

$$\frac{\partial}{\partial t}\overline{\theta^2} + \frac{\partial}{\partial z}\overline{w\theta^2} = -2\frac{\partial T}{\partial z}\overline{w\theta} + \chi\frac{\partial^2\overline{\theta^2}}{\partial z^2} - 2\epsilon_\theta + C^\theta \; ; \tag{82}$$

Turbulent kinetic energy in the z-direction $\frac{1}{2}\overline{w^2}$:

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{w^2} + \frac{\partial}{\partial z}\left(\frac{1}{2}\overline{w^3} + \overline{pw}\right) = -c_4^*\tau^{-1}\left(\overline{w^2} - \frac{1}{3}\overline{q^2}\right) + \frac{1}{3}(1+2\beta_5)g\alpha\overline{\theta w} - \frac{1}{3}\epsilon + \frac{1}{2}C_{33} - P_w^{nl} \; ; \tag{83}$$

Turbulent kinetic energy $e = \frac{1}{2}\overline{q^2}$:

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{q^2} + \frac{\partial}{\partial z}\left(\frac{1}{2}\overline{q^2 w} + \overline{pw}\right) = g\alpha\overline{w\theta} - \epsilon + \frac{1}{2}C_{ii} \,. \tag{84}$$

The terms C_3, C^θ, C_{ii}, and C_{33} are given by equations (42)–(46); $\overline{pw}$ and $\overline{p\theta}$ are given by equations (47) and (43); the third-order moments are given by equations (71)–(76). The nonlinear contributions to the pressure terms are given by equations (53) and (55). They are (b_{ij} is given by eq. [12c]).

$$(g\alpha)^{-1}eP_\theta^{nl} = \tfrac{1}{2}b_{33}(\gamma_2 + \tfrac{1}{2}\gamma_5 b_{33}e^{-1})\overline{\theta^2} + (\gamma_3 + \gamma_4 b_{33}e^{-1})(\overline{w\theta})^2 \,, \tag{85}$$

$$2(g\alpha)^{-1}eP_w^{nl} = -b_{33}\overline{\theta w}(1 + \beta_5 - \tfrac{3}{2}\beta_5 b_{33}e^{-1}) \,. \tag{86}$$

The functions γ's, β's, and the constants are given in Appendix A. To solve equations (81)–(84), one further needs the temperature gradient $\partial T/\partial z$ and the dissipation rates ϵ and ϵ_θ, to which we now turn.

As shown in Paper I, § 12, the local limit of equations (81)–(84) yields the MLT results (Bohm-Vitense 1958; Cox and Giuli 1968).

10. THE TEMPERATURE PROFILE: NEW FLUX CONSERVATION LAW

The true temperature gradient is obtained by solving the flux conservation law which we now derive. Using the definitions

$$F_c = c_p\rho\overline{w\theta} \,, \qquad F_r = -K\frac{\partial T}{\partial z} \,, \tag{87}$$

we have, from integrating (26),

$$F_{\text{ext}} + F_d = F_r + F_c - \int c_p\rho N_3(z)dz \,. \tag{88}$$

The new term F_d:

$$F_d \equiv \int \epsilon(z)\rho\, dz \tag{89}$$

represents the contribution to the external flux due to the generation of entropy, and thus of temperatue, arising from the (rate of) dissipation due to frictional forces. While this term is usually neglected (e.g., Zahn 1991), it must be kept for a consistent derivation of the full conservation law. As for the term N_3, we use equation (30) and neglect higher order terms to derive

$$c_p\rho N_3 = -\left(\frac{\partial}{\partial z} + \frac{1}{H_p}\right)F_p - (1-\gamma^{-1})\frac{1}{H_p}F_c \,, \tag{90}$$

where F_p is the *pressure flux*

$$F_p \equiv -\overline{pw} \,. \tag{91}$$

Thus, equation (88) becomes

$$F_{\text{ext}} + F_d = F_r + F_c(1+\eta_1) + F_p(1+\eta_2) \,, \tag{92}$$

where

$$\eta_1 F_c \equiv \int (1-\gamma^{-1})H_p^{-1}F_c(z)dz \,, \qquad \eta_2 F_p \equiv \int H_p^{-1}F_p(z)dz \tag{93}$$

are the contributions due to the non-thin layer approximation. To exhibit explicitly the turbulent kinetic energy flux

$$F_{\mathrm{KE}} = \tfrac{1}{2}\rho\overline{q^2 w}\,, \tag{94}$$

we first integrate the kinetic energy equation (84) thereby obtaining (in the stationary case and keeping the terms of highest order)

$$F_{\mathrm{KE}} + F_{\mathrm{d}} = \eta_1 F_c + F_p(1 + \eta_2)\,. \tag{95}$$

Combining equation (95) with equation (92) we obtain

$$F_{\mathrm{ext}} = F_r + F_c + F_{\mathrm{KE}}\,, \tag{96}$$

which shows that the total external flux is transported by radiation, convection, and turbulent kinetic energy, while in most astrophysical calculations the flux conservation is taken to be

$$F_{\mathrm{ext}} = F_r + F_c\,, \tag{97}$$

whereas it is known that in sunlike stars F_{KE} can be up to 50% of the total flux (Chan & Sofia 1989).

Finally, the expression for $\partial T/\partial z$ needed in equations (81) and (82) is determined from equation (96) which gives

$$\frac{\partial T}{\partial z} = \frac{\partial T_r}{\partial z} + \chi^{-1}\overline{w\theta} + K^{-1}F_{\mathrm{KE}}\,, \tag{98}$$

where we have introduced the customary fictitious temperature gradient $\partial T_r/\partial z$ via the relation $F_{\mathrm{ext}} \equiv \mathrm{KTH}_p^{-1}\nabla_r$, where ∇_r is the gradient as if the whole external flux were transported by radiation.

11. DISSIPATION: LOCAL AND NONLOCAL MODELS

The solution of the basic equations (81)–(84) requires the knowledge of both ϵ and ϵ_θ. We suggest two models. The *local model* is represented by equations (9), where e and $\overline{\theta^2}$ are obtained from solving equations (84) and (82). However, the mixing length l would have to be prescribed using empirical expressions, an approach used by several authors (Xiong 1986, Umezu 1992; Unno et al. 1985; Unno & Kondo 1989). For example, Xiong finds that the extent of the OV is given by

$$d = 1.4l\,, \qquad l = c_1 H_p\,. \tag{99}$$

Nonlocal model.—In this case, ϵ is the solution of the differential equation (Paper I, eqs. [46]–[37f]; Zeman 1992, private communication)

$$\tau\frac{\partial}{\partial z}(\tau^{-1}\overline{q^2 w}) = 2a_1 g\alpha\overline{w\theta} - a_2\epsilon + a_3 N\overline{w^2}\,, \tag{100a}$$

where $\tau = 2e/\epsilon$, $a_1 = 1.44$, $a_2 = 3.8$ and (for the definition of β, see eq. [48b])

$$N^2 = |g\alpha\beta|\,. \tag{100b}$$

Furthermore, in the case of unstable stratification $a_3 = 0$, while for stable stratification $a_3 = 0.1$. Equation (100a) can be integrated to yield (C is a constant of integration with dimensions)

$$\epsilon(z) = F_{\mathrm{KE}}\frac{\Phi(z)}{C + \int \Phi(z')dz'}\,, \tag{101a}$$

where

$$\Phi(z) \equiv \tfrac{1}{2}a_2\, e(z)\psi(z)F_{\mathrm{KE}}^{-2}\,, \tag{101b}$$

$$\psi(z) \equiv \exp\int (a_1 g\alpha\overline{w\theta} + \tfrac{1}{2}a_3 N\overline{w^2})F_{\mathrm{KE}}^{-1}\,dz'\,. \tag{101c}$$

The presence of the integrals is a clear indication of the nonlocal character of the expression for $\epsilon(z)$. As far as ϵ_θ is concerned, we suggest the use of equation (9a), as recently done by Hanjalic & Vasic (1993).

The ratio of ϵ computed nonlocally via equation (101) to the ϵ calculated via the local expression

$$\epsilon = \frac{e^{3/2}}{l_\epsilon}\,, \tag{102}$$

is shown in Figure 3. The ratio is relatively close to unity in the highly unstable convective layer while it differs greatly from unity as one approaches the inversion layer (stable stratification) where the local expression underestimates the dissipation.

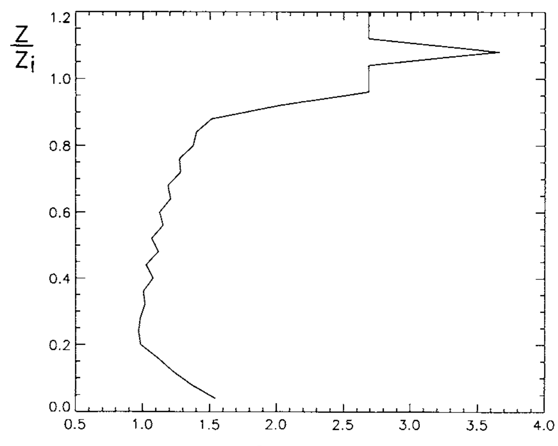

FIG. 3.—The ratio of the dissipation rate ϵ computed nonlocally to the value computed locally, eqs. (101) and (102), for the case of a convective planetary layer. As one can see, within the main convective region, the ratio is of order unity but the nonlocal ϵ exceeds the local value in the stable region.

12. HYDROSTATIC EQUILIBRIUM EQUATION

Using equation (24) in the stationary limit and equation (28), in which we neglect the fourth-order moment as well as equation (35), we obtain the new hydrostatic equilibrium equation

$$\frac{\partial}{\partial z}(P + p_t) = -g\rho(1 + x) , \tag{103}$$

$$p_t = \rho\overline{w^2} , \quad x = f_1 \tau^{-1} g^{-1} \alpha \overline{w\theta} + \gamma_1 \alpha^2 \overline{\theta^2} . \tag{104}$$

Turbulence has therefore two effects: the kinematic part contributes a turbulent pressure p_t, while buoyancy contributes to gravity via the convective flux $\overline{w\theta}$ and the temperature variance $\overline{\theta^2}$, equation (52).

13. OVERSHOOTING: PREVIOUS MODELS AS PARTICULAR CASES OF THE PRESENT EQUATIONS

As shown above, the complete nonlocal model requires the solution of *five* differential equations for the functions $\overline{w\theta}$, $\overline{q^2}$, $\overline{\theta^2}$, $\overline{w^2}$, and ϵ. The majority of the models proposed thus far employ only *one* differential equation for one turbulent variable, usually chosen to be $\overline{w^2}$ or, less frequently, $\overline{w\theta}$. The most complete treatment to date (Xiong 1981, 1985a, b, 1986) solves *three* differential equations for $\overline{w\theta}$, $\overline{w^2}$, and $\overline{\theta^2}$, while both dissipations ϵ and ϵ_θ are treated locally.

The dissipation ϵ.—Most authors have assumed $\epsilon = 0$ (Roxburgh 1978, 1989; Shaviv & Salpeter 1973; Bressan et al. 1981; Bertelli et al. 1986; Zahn 1991). The most commonly adduced argument is that ϵ is proportional to ν, while the fact that it is at the same time also proportional to a divergent mean-squared vorticity is overlooked. We have already discussed the fact that the value of ϵ *is not governed by the small scales or much less by viscosity, but by the large eddies which are the repository of most of the energy. What is determined by viscosity is the scale at which dissipation occurs.* This occurs because the nonlinear interactions that distribute the energy input to the whole spectrum of eddies conserve energy, and so the entire energy fed into the system is cascaded unaltered to the smaller scales where it is dissipated by molecular processes. Turbulence modeling over the years has shown that ϵ is one of the most difficult variables to compute and that its determination requires solving the whole problem, that is, equations (81)–(84). It is therefore not surprising if ϵ is such a critical variable; see equation (118) below.

Furthermore, numerical simulations covering the pressure range $0 \leq \ln P \leq 8$ (Chan & Gigas 1992, Fig. 2), show that ϵ is *smaller* than buoyancy work in the region $0 < \ln P < 4$, but *higher* than buoyancy work in the region $4 \leq \ln P < 8$, so that the integrated quantities satisfy equation (5c). In Figure 2 we show the ratio of the local values of the dissipation ϵ to the production ($\equiv g\alpha\overline{w\theta}$) terms, i.e., the two terms on the right-hand side of equation (118a), as derived from recent large eddy simulation of turbulent convection (K. L. Chan, private communication).

Thus, one must conclude that the assumption $\epsilon = 0$ *is not an approximation but a violation of an energy conservation law.*

Some authors have attempted to include the effect of ϵ in their model. For example, Maeder (1975) and Doom (1985) have suggested that ϵ be taken as

$$\rho\epsilon = -(1 - \gamma)T^{-1}\frac{\partial T}{\partial r}F_c , \tag{105}$$

with $0 \leq \gamma \leq 1$. Numerical simulations show that γ is hardly a constant and a constant γ would not satisfy the conservation law (5c). Further comments can be found in Umezu (1992). Other authors (Shaviv and Chitre 1968; Umezu 1991, 1992; Kuhfuss 1986, 1987) have used the local expression

$$\epsilon = D\frac{e^{3/2}}{l} , \tag{106}$$

with a constant D, interpreted as a drag coefficient.

Shaviv & Chitre (*1968*); *Umezu* (*1991, 1992*).—These authors suggest a nonlocal equation for the velocity field $w = (\overline{w^2})^{1/2}$ of the form

$$w \frac{\partial w^2}{\partial z} = g\alpha\overline{w\theta} - D\frac{w^3}{l}, \tag{107}$$

where $0.1 < D < 1$.

The question now arises: can equation (107) be derived from equation (83)? First, let us compare the left-hand sides of the two equations. Adopting the diffusion approximation, we have from equation (72)

$$\overline{w^3} = -|B_{21}|\tau w^2 \frac{\partial w^2}{\partial z} \equiv -\nu_t \frac{\partial w^2}{\partial z}, \tag{108}$$

where we have used the fact that B_{21} is negative (Appendix B). Since the pressure term $\overline{pw}$ is a third-order moment, it can be considered included in equation (108). If $\nu_t = lw$, then equation (83), neglecting the anisotropy and non-Boussinesq terms, becomes

$$fw\frac{\partial w^2}{\partial z} - \nu_t \frac{\partial^2 w^2}{\partial z^2} = -2c_4^* \tau^{-1}\left(\overline{w^2} - \frac{1}{3}\overline{q^2}\right) + \frac{2}{3}(1 + 2\beta_5)g\alpha\overline{\theta w} - \frac{2}{3}\epsilon, \tag{109a}$$

where ($\xi = z/l$)

$$f = -\frac{1}{\nu_t}\frac{\partial \nu_t}{\partial \xi}. \tag{109b}$$

The left-hand side of equation (109a) is quite different from that of equation (107) since f is not necessarily a positive constant and the curvature term is not necessarily zero. It seems that we have to consider an even *lower order than the diffusion approximation*, namely

$$\overline{w^3} \approx (\overline{w^2})^{3/2}, \tag{110}$$

so that equation (83) becomes

$$\frac{3}{2} w \frac{\partial w^2}{\partial z} = -2c_4^* \tau^{-1}\left(\overline{w^2} - \frac{1}{3}\overline{q^2}\right) + \frac{2}{3}(1 + 2\beta_5)g\alpha\overline{\theta w} - \frac{2}{3}\epsilon, \tag{111}$$

which is more similar to equation (107). However, how reliable is equation (110)? In Figure 4 we show, for the case of the strongly convective planetary boundary layer, that approximation (110) is satisfied in the main part of the convective layer but that it fails as one nears the region of stable stratification, which is precisely the region of interest.

Consider now the right-hand sides of equations (111) and (107). The first term in equation (111), representing anisotropy, is absent in equation (107), implying that turbulence has been assumed isotropic. However, we have already seen that such an approximation may be valid in the main body of the convective region but that it ceases to be so as one approaches the region of OV which is the seat of large anisotropies. Figure 5 shows this fact for the case of the convective planetary boundary layer.

As for the dissipation ϵ, equation (107) uses the local expression (106) with a constant D. However, we have seen from equation (66) that D can be taken constant in the main convective region, but not so in the OV regions where it is actually *less* than the

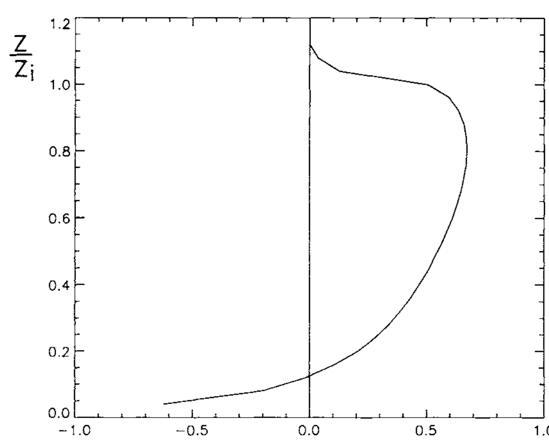

FIG. 4

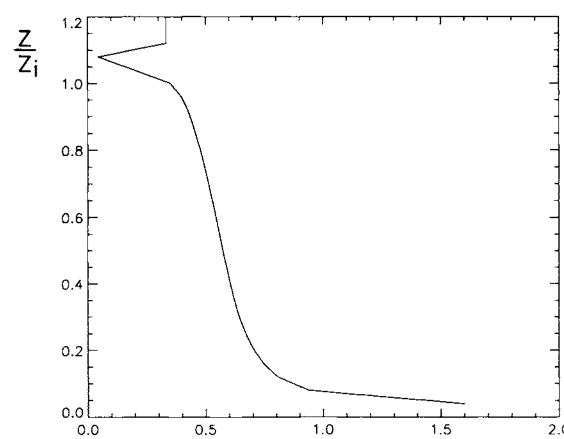

FIG. 5

FIG. 4.—The representation of the third-order moment $\overline{w^3}$ via $(\overline{w^2})^{3/2}$ seems reasonable within the main convective region but fails in the stable region where $\overline{w^3}$ goes to zero faster than $(\overline{w^2})^{3/2}$. The result is valid for a convective boundary layer.

FIG. 5.—The value of the ratio $q^2/3w^2$ clearly shows the high degree of anisotropy within a strongly convective boundary layer and especially in the stable region

convective region (see eq. [67]). Since the authors cited above have used the same velocity independent D throughout the entire convective regime, they have overestimated ϵ and thus underestimated the extent of OV.

Last, consider the treatment of the convective flux $\overline{w\theta}$. In principle, this requires that one consider another equation, (81). Since this is not done, the flux is written as

$$\overline{w\theta} = C_{w\theta}(\overline{w^2})^{1/2}(\overline{\theta^2})^{1/2} . \tag{112}$$

Using again the convective planetary boundary layer, we present in Figure 6 the correlation coefficient $C_{w\theta}$ which is constant only in the main convective region while changing rapidly and becoming negative as one approaches the OV region. Umezu (1991, 1992) has taken a variable $C_{w\theta}$ while Shaviv & Chitre (1968) have not.

This type of model has recently been used to study OV in massive stars (Umezu 1991, 1992) and in the Sun (Antia and Chitre 1993). From the last model one deduces that as D decreases ($D = 1$, 0.5, 0.3), the OV distance d increases ($d/H_p = 0.17$, 0.30, 0.40) (H. M. Antia, private communication 1992).

Bressan et al. (1981) and Bertelli et al. (1986).—To account for nonlocality, these authors also employ a differential equation for the velocity field w. In the chemically homogeneous case, their equation (6) is of the form of equation (107) and thus, all the previous discussion applies here too. There are, however, two important differences: dissipation is neglected, $\epsilon = 0$, which alone overestimates the OV region, and the convective flux $\overline{w\theta}$ is not determined from equation (112) but from assuming that the real temperature gradient ∇ is equal to the adiabatic value in both the convective and overshooting regions. F_c is then computed from the flux conservation law. Renzini (1987) is of the opinion that the latter approximation is the cause of the what he terms a "dramatically large overshooting" obtained by these authors. Since the inclusion of dissipation would *decrease* the extent of OV, it is not clear to us at this point how much the approximation $\nabla = \nabla_{ad}$ alone is responsible for (we shall return to this point when we discuss Roxburgh's model).

Kuhfuss (1986, 1987).—This model is more complex than the previous ones in that the kinetic energy equation includes the third-order moment under the diffusive approximation, equation (108), with the kinetic energy e substituted for $\frac{1}{2}w^2$. However, it has recently been shown (Canuto et al. 1992) that the down-gradient formula approximates very poorly the true third-order moments, as indicated in the case of the flux of kinetic energy $\frac{1}{2}\overline{q^2w}$ by Figure 7. Application of the model to chemically homogeneous ZAMS stars can be found in the 1987 paper.

Ulrich (1976) and Gough (1976).—Rather than a nonlocal equation for w, these authors have suggested a nonlocal equation for the convective flux $F_c = \overline{w\theta}$ ($c_p\,\rho = 1$). The proposed model equation is of the form

$$\frac{\partial^2}{\partial\zeta^2} F_c + p_0 \frac{\partial}{\partial\zeta} F_c - p_1 F_c = \left[(1-a)\frac{\partial^2}{\partial\zeta^2} + p_2\frac{\partial}{\partial\zeta} - p_1\right] F_c^{\rm MLT} . \tag{113a}$$

Here $\zeta = z/H_p$ and $F_c^{\rm MLT}$ is the local MLT expression. The constants p's are defined as

$$p_0 = (\alpha' - \alpha'')p_1 , \qquad p_1 = (\alpha'\alpha'')^{-1} , \qquad p_2 = (\tfrac{3}{2} - a)p_0 , \tag{113b}$$

where the α's and a are free parameters. When $a = 1$, $\alpha'' = \alpha' \equiv \alpha_*$, equation (113a) reduces to Gough's (1976) equation (11.14)

$$\frac{\partial^2}{\partial\xi^2} F_c - p_1 F_c = -p_1 F_c^{\rm MLT} , \tag{113c}$$

with $\xi = z/\Lambda$, $\Lambda = \alpha H_p$, and $\alpha = 3^{1/2}\alpha_*$.

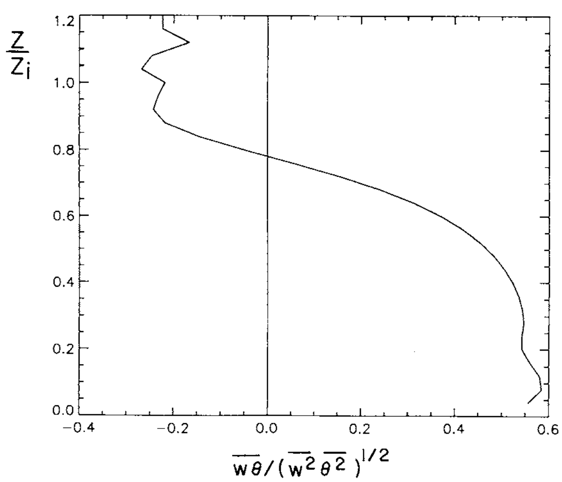

FIG. 6

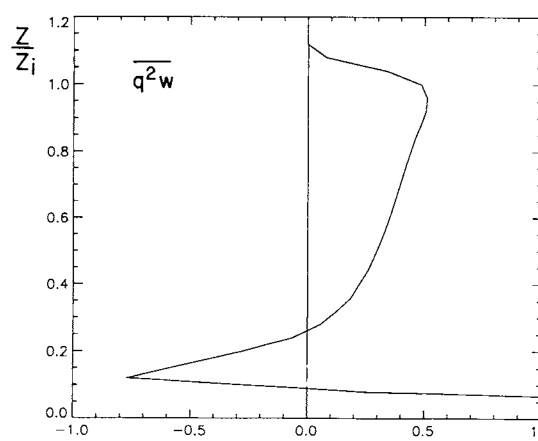

FIG. 7

FIG. 6.—The convective flux $\overline{w\theta}$ can be represented à la MLT, i.e., as the product of $(\overline{w^2})^{1/2}$ and $(\overline{\theta^2})^{1/2}$ only in a very narrow region of a convective boundary layer

FIG. 7.—The third-order moment $\overline{q^2w}$ is poorly represented by the down-gradient approximation. The curve shows the true value of the third-order moment, computed with the full equation (75), divided by the last term which represents the down-gradient only.

Can equation (113a) be derived from the flux equation (81)? First, consider the third-order moment $\overline{w^2\theta}$ in the diffusion approximation,

$$\overline{w^2\theta} = -\nu_t \frac{\partial}{\partial z} \overline{w\theta} , \qquad \nu_t = |A_1| , \tag{114}$$

where we have used the fact that the leading terms in A_1 are negative (Appendix B). Equation (81) then becomes (neglecting higher order terms)

$$\frac{\partial^2}{\partial z^2} F_c + p_0 \frac{\partial}{\partial z} F_c - p_1 F_c = -\nu_t^{-1}[\beta\overline{w^2} + (1-\gamma_1)g\alpha\overline{\theta^2}] , \tag{115a}$$

where

$$p_0 \equiv \frac{1}{\nu_t}\frac{\partial}{\partial z}\nu_t , \qquad p_1 \equiv f_1(\tau\nu_t)^{-1} . \tag{115b}$$

Note that the linear term $p_1 F_c$ originates from the pressure-temperature correlation Π_i^θ, equation (53).

Since there is no unique prescription of how to introduce the MLT convective flux, we shall do so by computing the right-hand side of equation (115a) in the local limit using the MLT expressions for $\overline{\theta^2}$ and $\overline{w^2}$, equations (73a)–(79), Paper I, together with equation (9a) to express $\overline{\theta^2}$ in terms of $\epsilon_\theta \equiv \beta\overline{w\theta}$. Equation (115) then becomes

$$\frac{\partial^2}{\partial \xi^2} F_c + \Lambda p_0 \frac{\partial}{\partial \xi} F_c - p_1 F_c = -p_3 F_c^{\rm MLT} , \tag{115c}$$

where $\xi = z/\Lambda$, $\Lambda = l/c_\epsilon$ and $8p_1 = f_1|A_{11}|^{-1}$, $p_3 = 4p_1(f_1 C)^{-1}[1 + (1-\gamma_1)\tau_\theta/\tau_\epsilon]$. Here, $\tau_\epsilon \equiv \tau$ and τ_θ correspond to the dissipation time scales of turbulent kinetic energy and temperature variance, respectively. The constant C, defined as $3c_6 c_4 C \equiv 2(2 + c_4 - 2c_5)$, is given in terms of the constants c's defined in Paper I. As an illustration, we employ the values suggested by equation (44d, Paper I) and Appendix B, namely $\gamma_1 = 7.5$, $\tau_\epsilon/\tau_\theta = 1.6$, $C = 0.3$, $|A_{11}| = \frac{1}{8}$, and obtain $p_1 = 7.8$ and $2p_3 = 5p_1$. Evidently, due to the different definitions of the scale lengths, one can in principle change these values but not the ratio p_3/p_1. Some considerations are in order.

a) Gough's equation (113*c*) does not include F_c' ($\equiv \partial F_c/\partial z$), while Ulrich's general case does, even though his model results do not include this term. Using equation (115b), we see that this is equivalent to assuming $p_0 = 0$ or, equivalently,

$$\nu_t \sim \text{constant} . \tag{116}$$

However, even on the basis of a simple $K - \epsilon$ model ($K = \frac{1}{2}q^2$) one would have instead

$$\nu_\tau \sim \frac{K^2}{\epsilon} \sim \frac{l^4 N^4}{g\alpha F_c} \sim g\alpha l^4 \beta^2 F_c^{-1} , \tag{117}$$

where $\epsilon = g\alpha\overline{w\theta}$.

b) We had to treat the right-hand side of equation (115a) with the local approximation (MLT). Physically, this means the following: a local theory interprets convection as a process in which density (or temperature) fluctuations feed velocity fluctuations, $\theta^2 \to w^2$ (see Fig. 1), which is true in the main convective regime; however, in the OV region, the opposite is true (Fig. 1): velocity fluctuations feed temperature fluctuations, and this reverse process is not captured by the MLT. Thus, the local treatment of right-hand side of equation (115a) does not account for the main physical process characterizing the OV region.

c) The down-gradient approximation (114), which may be acceptable in the main part of the convective region, fails as one nears the OV region, Figure 8.

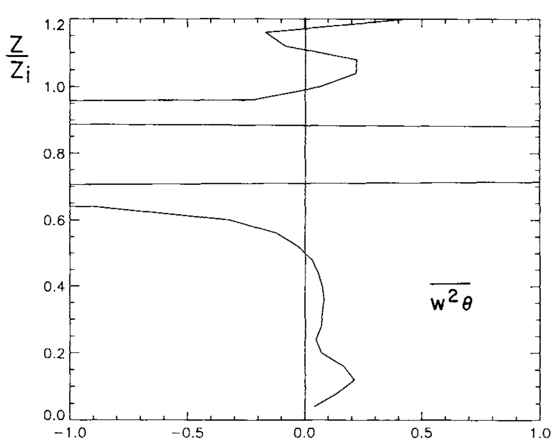

FIG. 8.—Same as in Fig. 7 for the third-order moment $\overline{w^2\theta}$

Roxburgh (1978, 1989) and Doom (1985).—The starting relation of this model is the equation for the turbulent kinetic energy $\frac{1}{2}\rho\overline{q^2}$, equation (84), which, neglecting C_{ii}, becomes

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\overline{q^2} + (1 - 2C_p^w)\frac{\partial}{\partial z}\frac{1}{2}\rho\overline{q^2 w} = -\frac{1}{T}\left(\frac{\partial T}{\partial z}\right)_{\rm ad} F_c - \rho\epsilon\,, \tag{118a}$$

whereas Roxburgh writes it as (see Doom 1985, eq. [3])

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\overline{q^2} + \frac{\partial}{\partial z}\frac{1}{2}\rho\overline{q^2 w} = -\frac{1}{T}\left(\frac{\partial T}{\partial z}\right) F_c\,. \tag{118b}$$

As one can seen, there are three differences between our equation and equation (118b): (1) in the present model, the pressure forces act to decrease the flux of turbulent kinetic energy $F_{\rm KE}$ by the factor $1 - 2C_p^w$. This should lead to a smaller OV region. This term is absent in equation (118b); (2) in equation (118a), we have, $\epsilon \neq 0$ and this again tends to *decrease* the extent of OV. In equation (118b), $\epsilon = 0$; (3) equation (118a) contains $(\partial T/\partial z)_{\rm ad}$ whereas equation (118b) contains $(\partial T/\partial z)$. Let us begin with this last point. The time evolution of the kinetic energy is derived from the basic Navier-Stokes equations which do not contain temperature gradients. Temperature enters only via the density variation in the gravity (buoyancy) term; using equation (23), the density difference is then transformed into a temperature difference ($\equiv \theta$) which, after being multiplied by a velocity to get the kinetic energy, produces the term $\overline{w\theta}$, i.e., F_c. Therefore, the combination of g and c_p gives rise to $(\partial T/\partial z)_{\rm ad}$ and not to $(\partial T/\partial z)$.

Physically, in a convective regime a temperature gradient sets up a convective flux not kinetic energy. In fact, $\partial T/\partial z$ appears as a source on the right-hand side of equation (81). In turn, the convective flux acts like a source for the kinetic energy which explains the presence of F_c in equation (118).

The $\partial T/\partial z$ term in equation (118b) presents a further problem. When $\partial T/\partial z$ is negative, F_c is positive (convectively unstable region), whereas when $\partial T/\partial z$ is positive, F_c is negative (stable stratification, OV region), Figure 1. Since the right-hand side of equation (118b) would be positive in either case, it would act like a *source*, which is contrary to the fact that in the OV region it must act like a *sink* since kinetic energy is converted to potential energy and thus must decrease. On the other hand, the right-hand side of equation (118a) is positive (source) when $F_c > 0$, and negative (sink) when $F_c < 0$, since the adiabatic gradient is always negative.

Equation (118a) can be transformed into a differential equation for the turbulent kinetic energy flux, $\frac{1}{2}\rho\overline{q^2 w}$. Eliminating F_c via the flux conservation law equation (96), one obtains the expression for $F_{\rm KE}$. Integrating the result between the extremes r_1 and r_2 of the convective plus OV regions, we obtain with an obvious change of notation so as to introduce the luminosity L,

$$\int_{r_1}^{r_2}\left[T^{-2}\left(\frac{\partial T}{\partial r}\right)_{\rm ad}(L_N - L_r) + 4\pi r^2 T^{-1}\rho\epsilon\right]\Phi(r)dr = 0\,, \tag{119a}$$

where

$$\ln\Phi(r) \equiv \int\frac{1}{T}\left[\left(\frac{\partial T}{\partial r}\right) - \frac{1}{\sigma}\left(\frac{\partial T}{\partial r}\right)_{\rm ad}\right]dr\,, \tag{119b}$$

with $\sigma \equiv 1\text{–}2C_p^w$. On the other hand, if we employ equation (118b) and carry out the same calculation, we obtain the Roxburgh criterion

$$\int_{r_1}^{r_2} T^{-2}\left(\frac{\partial T}{\partial r}\right)(L_N - L_r)dr = 0\,. \tag{119c}$$

Comparing equations (119a) and (119c) we see that Roxburgh's criterion is based on the following three approximations:

$$C_p^w = 0\,, \qquad \epsilon = 0\,, \qquad \frac{\partial T}{\partial r} = \left(\frac{\partial T}{\partial r}\right)_{\rm ad}\,, \tag{120}$$

that is, no-pressure forces, no dissipation, and $\nabla = \nabla_{\rm ad}$. As we have already discussed, $\epsilon = 0$ is not an approximation but a violation of the global energy conservation.

As for the last of equation (120), it is not required either by the mathematics or by the physics of the problem. For example, in the case of massive stars, equation (120) has not been confirmed by the most complete model presently available (Xiong 1985a, b, Fig. 3) which instead predicts that (cr = convective region; ov = overshooting region)

$$(\nabla - \nabla_{\rm ad})_{\rm ov} \approx 10^4(\nabla - \nabla_{\rm ad})_{\rm cr}\,, \tag{121}$$

indicating a very inefficient convection in the OV region. Furthermore, from a purely pragmatic point of view, $\nabla = \nabla_{\rm ad}$ biases the result by overestimating the extent of overshooting (Renzini 1987).

As for the first of equation (120), it is equivalent to neglecting a third-order moment which, however, has the effect of lowering the extent of overshooting.

In conclusion:

a) To solve the OV problem, one must use the basic equations (81)–(84), (96), and (100);

b) If one selects not to do so and considers only one of the basic equations, for example, the equation for the kinetic energy (84), as Roxburgh does, then;

c) The correct criterion for OV valid for an arbitrary temperature gradient, including pressure forces and dissipation, is equations (119a, b) not equation (119c);

d) In using equations (119a, b), one needs two critical ingredients, ∇ and ϵ. Assuming $\nabla = \nabla_{ad}$ is like guessing much of the solution. Guessing ϵ is even more difficult and taking $\epsilon = 0$ violates energy conservation;

e) With the critical ingredients ∇ and ϵ essentially undetermined, it seems fair to conclude that even the correct expressions (119a, b), *if considered alone*, are so severely incomplete to become useless, at least as a tool to predict the extent of the OV region.

Shaviv and Salpeter (1973).—This approach, which has been used by several authors (e.g., Maeder 1975; Pidatella & Stix 1986; Langer 1986; Skaley & Stix 1991), does not solve any of the dynamic equations (81)–(86): rather, it suggests an *Ansatz* of the form ($c_p \rho = 1$)

$$F_c = \overline{w(r)\theta(r)} \rightarrow w(r, r + \Lambda)\theta(r, r + \Lambda) , \tag{122a}$$

$$w^2(r, r + \Lambda) = \int_{r+\Lambda}^{r} g\alpha\theta(r', r' + \Lambda)dr' , \tag{122b}$$

$$\theta(r, r + \Lambda) = \int_{r+\Lambda}^{r} \beta(r')dr' , \tag{122c}$$

where $\beta = (T/H_p)(\nabla - \nabla_{ad})$. The first approximation is contained in equation (122a) where the ensemble average is split into the product of two quantities $w \equiv (\overline{w^2})^{1/2}$ and $\theta \equiv (\overline{\theta^2})^{1/2}$. One can further see that equation (122b) can be transformed into a differential form similar to equation (106) but *without the dissipation term*. The remarks made earlier about the physical relevance of the latter apply equally well in this case.

Xiong (1981, 1985a, b, 1986).—Xiong's work merits particular mention since in our opinion it is the most advanced formulation to date. Both Xiong's and our work have in common the general proposition to adopt the Reynolds stress decomposition method. However, they differ quite substantially in the treatment of several important physical processes (see the Introduction) which have been greatly clarified in the last decade of turbulence modeling. It may thus be instructive to list some of the most relevant differences. Specifically, we refer to Xiong (1985b, Appendix):

1. Kinetic energy: equation (A15) coincides with our equation (84) if we neglect the C_{ii} contribution. In fact, Xiong adopts the Boussinesq approximation while we do not.
2. Turbulence is assumed to be isotropic. For this reason, Xiong does not have two separate equations for $\overline{w^2}$ and $\overline{q^2}$. As one can see, in the limit $3\overline{w^2} = \overline{q^2}$, equation (83) becomes indistinguishable from equation (84). The assumption of isotropy is also responsible for the absence in Xiong's formulation of the P_w^{nl} terms in equation (83). However, as shown in Figure 4, anisotropy may be one of the key features of turbulence in the OV region and its neglect is not a priori justified.
3. The third-order moments: they are all treated with the diffusion approximation which is actually two approximations, the neglect of the gradients of other second-order moments as well as the contribution of buoyancy to the turbulent viscosity. As shown in Figure 6 (Canuto et al. 1992), the diffusion approximation yields incorrect results in the case of the Earth's convective boundary layer.
4. Temperature variance: if we compare equation (A17) with equation (82) we note that it lacks the C^θ term and that the third-order moment is treated as discussed in (3) above.
5. Convective flux: equation (A19) lacks the C_3 and P_θ^{nl} contributions; again, the third-order term is treated as discussed in (3) above.
6. Gravity waves: are not included since ϵ and ϵ_θ do not satisfy relations (61)–(62).
7. The *buoyancy range* characterized by a k^{-3} rather than a $k^{-5/3}$ spectrum is not accounted for because of the use of the Kolmogorov spectrum throughout.
8. Dissipation: this crucial variable is computed locally with equation (9). With a Kolmogorov constant of Ko = 1.6, one has

$$c_\epsilon = \pi\left(\frac{2}{3\ \mathrm{Ko}}\right)^{3/2} = 0.85 , \tag{123a}$$

which is the value employed by Xiong.

9. Nonlocality. The nonlocal nature of the model is wholly represented by the third-order moments which are treated as discussed in (3) above.
10. Mixing length and OV. The author employs the relation $l = c_1 H_p$. The extent of OV is found to be $d = 1.4l$, which, for $c_1 = 1$, $\frac{1}{3}$, yields values larger than the upper limit $d/H_p < 0.2$ arrived at by Stothers & Chin (1991). The use of a nonlocal model for ϵ would free the problem from the adjustable parameter c_1.
11. The model has been applied to massive stars by Xiong (1985a, b, 1986) and to the Sun by Unno et al. (1985) and Unno & Kondo (1989).

14. CONCLUSIONS

Zahn (1991) has lamented that astrophysicists neglect to confront their problems with those of colleagues who work in turbulence in other disciplines and who are often decades ahead. The advice, which we heeded before it was suggested, implicitly recognizes that geophysical turbulence has been treated for years with models that are considerably more complete than those used in stellar structure calculations. Today, stellar structure modelers have few alternatives: local models à la MLT and the not-yet-ready numerical simulations (LES, large eddy simulations).

If we plan to learn how other disciplines have advanced in the modeling of turbulence, the first thing to realize is that the choice between the two above alternatives overlooks entirely an important intermediate step that has played and continues to play, a major role in most turbulent studies, namely the second-order-closure model (SOC) based on the Reynolds stress approach. In fact, historically the development of turbulence modeling has passed through the following three phases:

$$\text{Phen. Relations} \to \text{SOC} \to \text{LES} . \tag{124}$$

What is missing in stellar turbulence is Phase 2, the use of SOC models based on the Reynolds splitting of the type discussed in this paper.

Jumping from phase 1 to phase 3 is neither presently feasible nor advisable. It is not feasible because the crucial ingredient in any LES code is the SGS, the sub-grid scale model to treat the unresolved scales, which is not yet available for physical situations in which convection, rather than shear, dominates. In fact, since a reliable SGS model will eventually result from an analysis similar to the one used in the SOC treatment presented in this paper, the SOC model is indispensable. It would not be advisable to skip the second phase because LES cannot be expected to be hooked up any time soon to already complex stellar structure codes. This last problem (which is perhaps only technical) mirrors again very closely what has occurred in the treatment of convective turbulence in the atmosphere. As we said earlier, SOC models have been in use since the early 70s. The appearance of LES dates back to the last 10 years and, as of today, there are four operational LES codes in existence for atmospheric turbulence. However, none of them has yet been linked to a global circulation model (the equivalent of a stellar structure code) essentially because of the time requirements involved. Rather, the four LES codes, viewed as experimental tools and thus run very parsimoniously, are used to generate results to improve the description of those difficult terms (e.g., third-order moments, pressure-velocity correlations, etc.) that the SOC approach must model.

It therefore seems to us that stellar structure calculations ought to enter phase 2, in which one abandons the phenomenological relations that have exhausted their fruitfulness and substitutes them with the SOC models. *The improvement in the quality of turbulence modeling would be substantial while the added numerical complexity to the stellar code is manageable.* In the meantime, one should use the mathematical and physical structure of the SOC to generate a reliable SGS model to be used in the LES codes presently available (Cattaneo et al. 1991; Stein and Nordlund 1989, Chan and Sofia 1989, Hurlburt, Toomre, & Massaguer 1984; Hossain and Mullan 1991; Chan and Gigas, 1992), so as to render them applicable to the very high Reynolds number and very low Prandtl number that characterize stellar interiors.

It is a pleasure to thank Prof. W. Unno, Prof. D. R. Xiong, Dr. R. Stothers, Mr. O. Schilling and Z. Wu for their careful reading of the manuscript and Drs. K. L. Chan, I. Mazzitelli, F. Minotti, and C. Ronchi for several useful discussions.

APPENDIX A

DIMENSIONLESS FUNCTIONS γ's AND CONSTANTS

1. The following equations are taken from (Shih & Shabbir 1992).

$$9\gamma_1 \equiv 6r^2 - 10 - \beta_5(r^2 - 1)^{-1}[18(r^2 + 1)rb + r^2(7 - 15r^2) + 36II - 10] , \tag{A1}$$

$$9(r^2 - 1)II\gamma_2 \equiv r^2(3II + 14) - (3II + \tfrac{7}{2}) - \tfrac{21}{2}r^4 - \beta_5[r^2(12II + 20) + 108IIrb + 108II^2 - 21II - 5 - 15r^4(1 + 3II)] , \tag{A2}$$

$$\gamma_3 \equiv -1 + \tfrac{1}{2}\beta_5(r^2 - 1)^{-1}(12rb - 5r^2 + 12II - 1) , \tag{A3}$$

$$\gamma_4 \equiv -\tfrac{3}{2}\beta_5 , \qquad 6II\gamma_5 \equiv 7 + \beta_5(36II - 10) , \tag{A4}$$

where

$$b_{ij} = \overline{u_i u_j} - \tfrac{2}{3}e\delta_{ij} , \qquad q^2 = 2e = \overline{u_i u_i} , \tag{A5}$$

$$r^2 \equiv \overline{\theta u_i}\,\overline{\theta u_i}(\overline{\theta^2}q^2)^{-1} , \qquad rb \equiv (\overline{\theta^2}q^2)^{-1}\overline{\theta u_i}\,\overline{\theta u_j}\,b_{ij} . \tag{A6}$$

Using data from a buoyant plume experiment, Shih & Shabbir (1992) have determined that the value of β_5 is approximately 0.6 which would correspond to $c_5 \equiv 1 - \beta_5 = 0.4$, a value close to 0.3 suggested in Paper I, equation (44d). In the same work, the authors have also shown that γ_1 is almost constant (~ 0.42), while $\gamma_{2,3,4,5}$ are all negative, with values ranging as follows:

$$1.91 < |\gamma_2| < 2.82 , \qquad 0.28 < |\gamma_3| < 0.7 , \qquad 0.81 < |\gamma_4| < 2.76 , \qquad 0.95 < |\gamma_5| < 3.15 . \tag{A7}$$

2. The function β's (Shih & Shabbir 1992):

$$\beta_5(6II - 10r^2 - 36IIrb) = -(12II + 7)r^2 , \qquad \beta_7 \equiv -7/6II + \beta_5(5/3II - 1) , \qquad -\beta_9 = \beta_5 + \beta_7 . \tag{A8}$$

3. The invariants II, III, and the function F:

$$-8e^2II \equiv b_{ij}b_{ij} , \qquad 24e^3III \equiv b_{ij}b_{jk}b_{ik} , \qquad F \equiv 1 + 9II + 27III . \tag{A10}$$

4. The c's:

$$c_4^* \equiv c_4 + 1 - F^{1/2} , \tag{A11}$$

$$c_4 = 1 + 6.22F^2(1 - F)^{3/4} . \tag{A12}$$

As for the other c's, we suggest the values discussed in Paper I. Furthermore,

$$C_p^w = C_p^\theta = \tfrac{1}{5} , \tag{A13}$$

$$f_1 = 7.5 . \tag{A14}$$

APPENDIX B

THE TURBULENT VISCOSITIES (DIFFUSIVITIES)

Each of the diffusivities $A_k, \ldots, F_k$, has the form exhibited in equations (71)–(76). Furthermore, each $A_{k1}, A_{k2}, \ldots, F_{k1}, F_{k2}$ has the form

$$\Delta A_{11} = A_{110} + A_{111}x + A_{112}x^2 + A_{113}x^3 , \tag{B1}$$

$$\Delta A_{12} = A_{120} + A_{121}x + A_{122}x^2 + A_{123}x^3 . \tag{B2}$$

The only exception is F_{21} which has an additional x^4. Since equations (71)–(76) arise from the inversion of a matrix, there is a denominator Δ whose structure is

$$\Delta = \Delta_0 + \Delta_1 x + \Delta_2 x^2 + \Delta_3 x^3 , \tag{B3}$$

where

$$\Delta_0 = 1.27\ 10^7 , \quad \Delta_1 = -4.71\ 10^5 , \quad \Delta_2 = 2.86\ 10^3 , \quad \Delta_3 = -4.6 . \tag{B4}$$

The dimensionless function x is defined as

$$x \equiv g\alpha\beta\tau^2 = (g/H_p)(\nabla - \nabla_{ad})\tau^2 , \tag{B5}$$

with τ given by equation (51) and where we have used

$$c_8 = 8 , \quad c_{10} = 4 , \quad c_{11} = \tfrac{1}{5} , \quad c_* = 0 , \quad c_2 = 1 . \tag{B6}$$

We have obtained the following numerical values ($E+04$ stands for 10^4, etc.):

$$A_{110} = -1.5838E+06 , \quad A_{111} = 2.8365E+04 , \quad A_{112} = -69.12 , \quad A_{113} = 0 ,$$
$$A_{120} = -1.3726E+05 , \quad A_{121} = 406.19 , \quad A_{122} = 0 , \quad A_{123} = 0 ,$$
$$A_{210} = 0 , \quad A_{211} = -1.3924E+05 , \quad A_{212} = 2.4576E+03 , \quad A_{213} = -5.76 ,$$
$$A_{220} = -7.9189E+05 , \quad A_{221} = 1.3926E+04 , \quad A_{222} = -30.72 , \quad A_{223} = 0 ,$$
$$A_{310} = -6.8631E+04 , \quad A_{311} = 203.09 , \quad A_{312} = 0 , \quad A_{313} = 0 ,$$
$$A_{320} = -2.0589E+04 , \quad A_{321} = 60.928 , \quad A_{322} = 0 , \quad A_{323} = 0 ,$$
$$A_{410} = 0 , \quad A_{411} = 5.1200E+03 , \quad A_{412} = -76.8 , \quad A_{413} = 0 ,$$
$$A_{420} = 0 , \quad A_{421} = 256.00 , \quad A_{422} = -3.84 , \quad A_{423} = 0 ,$$

$$B_{110} = -2.1299D+05 , \quad B_{111} = 3.8229E+03 , \quad B_{112} = -9.216 , \quad B_{113} = 0 ,$$
$$B_{120} = -1.8350E+04 , \quad B_{121} = 53.248 , \quad B_{122} = 0 , \quad B_{123} = 0 ,$$
$$B_{210} = -2.2128E+06 , \quad B_{211} = 6.3283E+04 , \quad B_{212} = -158.72 , \quad B_{213} = 0 ,$$
$$B_{220} = -1.1059E+05 , \quad B_{221} = 2.0173E+03 , \quad B_{222} = -4.608 , \quad B_{223} = 0 ,$$
$$B_{310} = -9.1750E+03 , \quad B_{311} = 26.624 , \quad B_{312} = 0 , \quad B_{313} = 0 ,$$
$$B_{320} = -2.7525E+03 , \quad B_{321} = 7.9872 , \quad B_{322} = 0 , \quad B_{323} = 0 ,$$
$$B_{410} = 8.1920E+04 , \quad B_{411} = -2.1163E+03 , \quad B_{412} = 0 , \quad B_{413} = 0 ,$$
$$B_{420} = 4.0960E+03 , \quad B_{421} = -105.81 , \quad B_{422} = 0 , \quad B_{423} = 0 .$$

$$C_{110} = 0\,, \quad C_{111} = -1.5838E+05\,, \quad C_{112} = 460.8\,, \quad C_{113} = 0\,,$$
$$C_{120} = -1.2670E+06\,, \quad C_{121} = 1.4336E+04\,, \quad C_{122} = -30.72\,, \quad C_{123} = 0\,,$$
$$C_{210} = 0\,, \quad C_{211} = 0\,, \quad C_{212} = -1.3824E+04\,, \quad C_{213} = 38.4\,,$$
$$C_{220} = 0\,, \quad C_{221} = -7.9189E+04\,, \quad C_{222} = 204.8\,, \quad C_{223} = 0\,,$$
$$C_{310} = -6.335E+05\,, \quad C_{311} = 7.1680E+03\,, \quad C_{312} = -15.36\,, \quad C_{313} = 0\,,$$
$$C_{320} = -1.9005E+05\,, \quad C_{321} = 2.1504E+03\,, \quad C_{322} = -4.608\,, \quad C_{323} = 0\,,$$
$$C_{410} = 0\,, \quad C_{411} = 0\,, \quad C_{412} = 512\,, \quad C_{413} = 0\,,$$
$$C_{420} = 0\,, \quad C_{421} = 0\,, \quad C_{422} = 25.6\,, \quad C_{423} = 0\,.$$

$$D_{110} = -1.5838E+06\,, \quad D_{111} = 2.6749E+04\,, \quad D_{112} = -84.48\,, \quad D_{113} = 0\,,$$
$$D_{120} = -1.5838E+05\,, \quad D_{121} = 672.43\,, \quad D_{122} = 0\,, \quad D_{123} = 0\,,$$
$$D_{210} = 0\,, \quad D_{211} = -8.1920E+04\,, \quad D_{212} = 388.27\,, \quad D_{213} = 0\,,$$
$$D_{220} = 0\,, \quad D_{221} = -1.3995E+04\,, \quad D_{222} = 61.44\,, \quad D_{223} = 0\,,$$
$$D_{310} = -7.9189E+04\,, \quad D_{311} = 336.21\,, \quad D_{312} = 0\,, \quad D_{313} = 0\,,$$
$$D_{320} = -2.3757E+04\,, \quad D_{321} = 100.86\,, \quad D_{322} = 0\,, \quad D_{323} = 0\,,$$
$$D_{410} = 0\,, \quad D_{411} = -4.0960E+04\,, \quad D_{412} = 1.5061E+03\,, \quad D_{413} = -5.76\,,$$
$$D_{420} = -7.9189E+05\,, \quad D_{421} = 2.7369E+04\,, \quad D_{422} = -103.68\,, \quad D_{423} = 0\,.$$

$$E_{110} = -1.9661E+05\,, \quad E_{111} = 3.4543E+03\,, \quad E_{112} = -9.216\,, \quad E_{113} = 0\,,$$
$$E_{120} = -1.7913E+04\,, \quad E_{121} = 61.44\,, \quad E_{122} = 0\,, \quad E_{123} = 0\,,$$
$$E_{210} = -1.3107E+06\,, \quad E_{211} = 3.3860E+04\,, \quad E_{212} = -76.8\,, \quad E_{213} = 0\,,$$
$$E_{220} = -6.5536E+04\,, \quad E_{221} = 573.44\,, \quad E_{222} = 0\,, \quad E_{223} = 0\,,$$
$$E_{310} = -8.9566E+03\,, \quad E_{311} = 30.72\,, \quad E_{312} = 0\,, \quad E_{313} = 0\,,$$
$$E_{320} = -2.6870E+03\,, \quad E_{321} = 9.216\,, \quad E_{322} = 0\,, \quad E_{323} = 0\,,$$
$$E_{410} = -6.5536E+05\,, \quad E_{411} = 2.3074E+04\,, \quad E_{412} = -92.16\,, \quad E_{413} = 0\,,$$
$$E_{420} = -3.2768E+04\,, \quad E_{421} = 1.1537E+03\,, \quad E_{422} = -4.608\,, \quad E_{423} = 0\,.$$

$$F_{110} = 0\,, \quad F_{111} = 0\,, \quad F_{112} = -5.9392E+04\,, \quad F_{113} = 172.8\,,$$
$$F_{120} = 0\,, \quad F_{121} = -4.7514E+05\,, \quad F_{122} = 5.3760E+03\,, \quad F_{123} = -11.52\,,$$
$$F_{210} = 0\,, \quad F_{211} = 0\,, \quad F_{212} = 0\,, \quad F_{213} = -5.1840E+03\,, \quad F_{214} = 14.40\,,$$
$$F_{220} = 0\,, \quad F_{221} = 0\,, \quad F_{222} = -2.9696E+04\,, \quad F_{223} = 76.8\,,$$
$$F_{310} = 0\,, \quad F_{311} = -2.3757E+05\,, \quad F_{312} = 2.6880E+03\,, \quad F_{313} = -5.76\,,$$
$$F_{320} = -4.7514E+06\,, \quad F_{321} = 1.0523E+05\,, \quad F_{322} = -267.52\,, \quad F_{323} = 0\,,$$
$$F_{410} = 0\,, \quad F_{411} = 0\,, \quad F_{412} = 0\,, \quad F_{413} = 192\,.$$

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